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THE THANKSGIVING MEETING IN LOS ANGELES

The three hundred eighteenth meeting of the American Mathematical Society was held at the University of California at Los Angeles on Saturday, December 1, 1934. There were about fifty-five persons in attendance, among whom were the following twenty-five members of the Society:

O. W. Albert, L. D. Ames, H. M. Bacon, Harry Bateman, Clifford Bell, E. T. Bell, H. F. Blichfeldt, P. H. Daus, Raymond Garver, H. E. Glazier, E. R. Hedrick, C. G. Jaeger, Glenn James, R. D. James, C. G. P. Kuschke, D. N. Lehmer, W. E. Mason, A. D. Michal, J. C. Montgomery, C. B. Morrey, G. E. F. Sherwood, D. V. Steed, F. C. Touton, S. E. Urner, E. R. Worthington.

A luncheon was served in Kerckhoff Hall, the Student Union building of the University.

In the absence of Associate Secretary Putnam, Professor P. H. Daus acted as secretary. The meeting was presided over in succession by Professors L. D. Ames, E. T. Bell, and H. F. Blichfeldt. Titles and cross-references to the abstracts of the papers read at this meeting follow below; papers whose abstract numbers are followed by the letter *t* were read by title. Messrs. Elconin, Highberg, Mersman, and Taylor were introduced by Professor A. D. Michal.

1. *Arithmetical theorems on Lucas functions and Tchebycheff polynomials*, by Professor E. T. Bell. (Abstract No. 40-11-374.)

2. *On the differentiability of the solutions of a class of regular two dimensional minimum problems*, by Dr. C. B. Morrey, Jr. (Abstract No. 40-11-376.)

3. *Waring's problem for rational numbers* (preliminary report), by Dr. R. D. James. (Abstract No. 40-11-375.)

4. *Postulates for groups and for commutative groups*, by Professor Raymond Garver. (Abstract No. 40-11-377.)

5. *Generalizations of the Mittag-Leffler theorem and allied theorems*, by Professor E. R. Hedrick. (Abstract No. 40-11-378.)

6. *The existence of a Hermitian inner product in complex vectorial spaces*, by Mr. Ivar Highberg. (Abstract No. 40-11-380.)

7. *A new type of abstract integral*, by Mr. W. A. Mersman. (Abstract No. 40-11-382.)

8. *Abstract euclidean spaces and semi-vector spaces* (preliminary report), by Professor A. D. Michal. (Abstract 40-11-383.)

9. *Multiple differentials and integrals of abstractly-valued functions*, by Mr. Victor Elconin. (Abstract No. 40-11-387.)

10. *Additions to the theory of integral invariants*, by Mr. A. E. Taylor. (Abstract No. 40-11-390.)

11. *Note on a theorem in abstract sets*, by Professor W. M. Whyburn. (Abstract 41-1-4-*t*.)

12. *Solutions of certain ordinary and partial differential equations by means of operators*, by Mr. John Biggerstaff. (Abstract No. 40-11-372-*t*.)

13. *On the sum of the rth derived series of the conjugate Fourier series*, by Professor A. F. Moursund. (Abstract No. 40-11-373-*t*.)

14. *The existence of rings in vectorial spaces*, by Mr. Ivar Highberg. (Abstract No. 40-11-379-*t*.)

15. *A generalization of the Lebesgue integral*, by Mr. W. A. Mersman. (Abstract No. 40-11-381-*t*.)

16. *Solutions of total differential equations in abstract vector spaces as functions of the initial parameters*, by Professor A. D. Michal and Mr. Victor Elconin. (Abstract No. 40-11-386-*t*.)

17. *Theorems on integral invariants of associated type*, by Professor A. D. Michal. (Abstract No. 40-11-384-*t*.)

18. *A theorem on total Fréchet differentials in abstract vector spaces*, by Professor A. D. Michal. (Abstract No. 40-11-385-*t*.)

19. *A reduced set of postulates for Hilbert space* (preliminary report), by Mr. A. E. Taylor. (Abstract No. 40-11-388-*t*.)

20. *Integral invariants and integrals of systems of first order differential equations*, by Mr. A. E. Taylor. (Abstract No. 40-11-389-*t*.)

21. *The completely symmetric self-dual curve of order eight*, by Dr. D. C. Duncan. (Abstract No. 41–1–3–*t*.)

22. *Completely symmetric elliptic self-dual curves of order* $4k$, by Dr. D. C. Duncan. (Abstract No. 41-1-2-*t*.)

23. *Bimagic squares*, by Professor D. N. Lehmer. (Abstract No. 41-1-1.)

P. H. DAUS,
Acting Secretary

THE THANKSGIVING MEETING IN LINCOLN

The three hundred seventeenth regular meeting of the Society was held at the University of Nebraska on Friday and Saturday, November 30–December 1, 1934. About sixty persons attended the meeting, among whom were the following twenty-nine members of the Society:

R. W. Babcock, Wealthy Babcock, M. A. Basoco, M. H. Beaty, F. L. Black, G. A. Bliss, W. C. Brenke, C. C. Camp, A. L. Candy, O. C. Collins, A. T. Craig, H. P. Doole, J. M. Earl, W. E. Ekman, M. G. Gaba, Emma Hyde, M. H. Ingraham, A. J. Kempner, H. I. Lane, U. G. Mitchell, T. A. Mossman, T. A. Pierce, B. L. Remick, D. H. Richert, L. L. Runge, E. B. Stouffer, J. S. Turner, W. G. Warnock, M. A. Wilder.

On Friday afternoon Professor A. J. Kempner of the University of Colorado gave an address entitled *On complex roots of algebraic equations*. On Saturday morning Professor G. A. Bliss of the University of Chicago spoke to the Society on *The problem of Bolza in the calculus of variations*.

On Friday evening a dinner was held at the Hotel Cornhusker. Thirty-nine mathematicians and their guests attended. Professor M. G. Gaba of the University of Nebraska acted as toastmaster, and he first called upon Professor W. C. Brenke who welcomed the visitors to the University of Nebraska. Professor Bliss was then called upon and he spoke concerning various recent experiments in education. Professor M. H. Ingraham of the University of Wisconsin talked on the action which had been taken at Wisconsin regarding mathematics as an entrance requirement. Dean E. B. Stouffer of the University of Kansas then spoke on the work of the committee of the Mathematical Association which is making a study of the training of teachers for college work and certain problems which might be best solved by a modification of the present work for the doctor's degree. Dean R. W. Babcock of Kansas State College pointed out the value of the smaller meetings of the Society which allow members living farther away from the places of frequent meetings to meet each other and talk about mathematics. Professor Kempner discussed the problems which have arisen from the rating by a national committee of various

universities as to their qualifications for giving the training leading to a doctor's degree.

Professors Brenke and Bliss presided at the meeting on Friday afternoon, and Professors Stouffer and Kempner presided on Saturday morning.

The titles of papers read at the meeting follow. Those whose abstract numbers are followed by *t* were read by title. Mr. Cowgill was introduced by Professor Brenke, Miss Humphreys and Mr. Ghent by Professor Dickson, Dr. Robertson by Professor Bliss, Dr. Anderson by Professor Holl, and Mr. Harper by Professor Rietz.

1. *On the summability of a class of series of Jacobi polynomials*, by Mr. A. P. Cowgill. (Abstract No. 40-11-362.)

2. *On certain systems of polynomials*, by Professor M. A. Basoco. (Abstract No. 40-11-367.)

3. *Arithmetized expansions for certain pseudo-periodic functions*, by Professor M. A. Basoco. (Abstract No. 40-11-369.)

4. *A generalization of Chevilliet's formula*, by Professor C. C. Camp. (Abstract No. 40-11-364.)

5. *Note on numerical evaluation of double series*, by Professor C. C. Camp. (Abstract No. 40-11-365.)

6. *Involutorial simple algebras and real Riemann matrices*, by Professor A. A. Albert. (Abstract No. 40-11-363-*t*.)

7. *The representation of integers as sums of values of quartic polynomials* (preliminary report), by Miss M. G. Humphreys. (Abstract No. 40-11-310-*t*.)

8. *Operational identities on point sets*, by Mr. Garrett Birkhoff. (Abstract No. 40-11-337-*t*.)

9. *The arithmetized expansions for certain doubly periodic functions of the third kind*, by Mr. G. D. Nichols. (Abstract No. 40-11-358-*t*.)

10. *A generalization of Waring's theorem* (preliminary report), by Mr. K. S. Ghent. (Abstract No. 40-11-359-*t*.)

11. *Stieltjes integral for functions convex in one direction*, by Dr. M. S. Robertson (National Research Fellow). (Abstract No. 40-11-360-*t*.)

12. *On a certain identity due to Hermite*, by Professor M. A. Basoco. (Abstract No. 40-11-368-*t*.)

13. *Further theta expansions useful in arithmetic*, by Professor

M. A. Basoco and Professor E. T. Bell. (Abstract No. 40-11-370-*t*.)

14. *Linear derivative inequalities*, by Professor H. J. Ettlinger. (Abstract No. 40-11-355-*t*.)

15. *On the Bernoulli distribution*, by Dr. Solomon Kullback. (Abstract No. 40-11-338-*t*.)

16. *On the multinomial distribution*, by Dr. Solomon Kullback. (Abstract No. 40-11-357-*t*.)

17. *Statics of special types of homogeneous elastic slabs with variable thickness*, by Dr. E. W. Anderson. (Abstract No. 40-11-371-*t*.)

18. *On the distribution of a certain product moment*, by Professor A. T. Craig. (Abstract No. 40-11-361.)

19. *On a function to represent infantile mortality*, by Mr. F. S. Harper. (Abstract No. 40-11-356.)

20. *Finite geometries* (preliminary report), by Professor M. G. Gaba. (Abstract No. 40-11-366.)

M. H. INGRAHAM,
Associate Secretary

THE ANNUAL MEETING IN PITTSBURGH

The forty-first Annual Meeting of the American Mathematical Society was held in Pittsburgh, Pennsylvania, from Thursday to Monday, December 27–31, 1934. The University of Pittsburgh and the Carnegie Institute of Technology acted as hosts. The scientific sessions of the Society were held in the Administration Building of the Carnegie Institute of Technology, while the informal New Year's Eve Party was held in the new Cathedral of Learning of the University of Pittsburgh. The arrangements made by the local committee were excellent, and the meeting proved an extremely interesting and pleasant one. At the session on Monday morning, on motion of Professor R. E. Langer, the Society passed a vote of thanks to the local committee and the inviting institutions for their generous hospitality.

As there was no quorum present at the meeting of the Board of Trustees, which had been called for Thursday, December 27, adjournment was taken and a postponed meeting was held in New York on Wednesday evening, January 2. The Council held a meeting on Thursday evening at 7:30 P.M.

The scientific meetings were opened with a general session on Thursday morning. On Thursday afternoon the Society met in three sections, one devoted to Analysis, the second to Algebra and Geometry, and the third to Mathematical Statistics and Applied Mathematics. On Friday morning there were two sectional meetings, one for Analysis and Topology and the other for Algebra.

At a general session on Friday afternoon, Professor A. B. Coble delivered his retiring address as President of the Society. His subject was *The geometry of the Weddle manifold* W_p. Later in the afternoon Professor Albert Einstein of the Institute for Advanced Study delivered the eleventh Josiah Willard Gibbs Lecture, entitled *An elementary proof of the theorem concerning the equivalence of mass and energy*. Both of these addresses will appear in early numbers of the Bulletin.

On Friday evening there was a joint session of the American Mathematical Society, Sections A and K of the American

Association for the Advancement of Science, the Mathematical Association of America, and the Econometric Society. A symposium was held on The Nature and Limitations of Statistical Proof. Professor E. B. Wilson spoke on *What is a proof?*, Professor C. F. Roos on *What do time series correlation coefficients show?*, Professor H. T. Davis on *Statistical proofs of periodicity in economic series*, and Mr. Max Sasuly on *Practical difficulties in proving statistical relationships*. Mr. Carl Snyder, Vice-President of Section K, presided.

A general session was held on Saturday morning. The afternoon was devoted to a symposium on Group Theory and Quantum Mechanics. This was a joint session of the Society with the American Physical Society. Professor E. P. Wigner spoke on *Symmetry relations in various physical problems*, Professor J. H. Van Vleck on *Some applications of group theory to non-relativistic problems*, and Professor Gregory Breit on *Some applications of group theory to Dirac's relativistic theory*. Professor J. von Neumann, who was to have spoken on *Representations and ray representations in quantum mechanics*, was unfortunately unable to be present due to illness. Professor H. P. Robertson presided.

On Monday morning there was a joint meeting with the Mathematical Association of America and Section A of the American Association for the Advancement of Science. Professor C. N. Moore delivered his retiring address as Vice-President of Section A. His subject was *Mathematics and science*. Professor Arnold Dresden also gave his retiring address as President of the Mathematical Association, speaking on the subject *A program for mathematics*. Professor H. L. Rietz presided at this session.

The banquet of the mathematical organizations was held on Saturday evening in the Georgian Room of the Hotel Webster Hall. About two hundred and ninety persons attended. Professor Dunham Jackson acted as toastmaster and he called on Professors G. D. Birkhoff, L. L. Dines, and Solomon Lefschetz.

The Annual Business Meeting of the Society was held on Friday afternoon. Dr. A. B. Brown, Dr. J. L. Doob, Dr. J. S. Frame, and Professor J. B. Rosenbach were appointed tellers for the annual election. It was reported that 190 ballots had been cast and that the following officers had been elected:

President, Professor Solomon Lefschetz.

Vice-President, Professor Harry Bateman.

Associate Secretary, Professor J. R. Kline.

Member of the Editorial Committee of the Bulletin, Professor W. R. Longley.

Member of the Editorial Committee of the Transactions, Professor J. D. Tamarkin.

Member of the Editorial Committee of the Colloquium Publications, Professor R. L. Moore.

Members of the Board of Trustees, Professor W. B. Fite, Dr. Robert Henderson, Professors W. R. Longley and G. W. Mullins, and Dean R. G. D. Richardson.

Members of the Council, Professors A. A. Albert, Harold Hotelling, R. E. Langer, D. V. Widder, and R. L. Wilder.

The following eleven institutions of learning and mathematical groups were elected to institutional membership in the Society:

Amherst College;
Columbia University;
Department of Mathematics, University of Nebraska;
Department of Mathematics, Wellesley College;
Division of Mathematics, Harvard University;
Duke University;
McGill University;
Members of Department of Mathematics, College of the City of New York;
Smith College;
Tulane University;
Wesleyan University.

Announcement was made of the election of the following persons to membership in the Society:

Mr. Gilbert F. Boeker, College of the City of New York;
Dr. Josephine Hughes Chanler, University of Illinois;
Professor Richard Courant, New York University;
Major Allen Parker Cowgill (retired), Lincoln, Nebraska;
Miss Carolyn Eisele, Hunter College;
Professor Emil Erno Heimann, East Central State Teachers College, Ada, Oklahoma;
Mr. Fritz Herzog, New York City;
Miss Evelyn M. Hull, New York City;
Dr. John C. Knipp, University of Pittsburgh;
Professor Emmy Noether, Bryn Mawr College;
Professor Ruth M. Peters, Judson College, Marion, Alabama;
Professor Hans A. Rademacher, University of Pennsylvania;

Mr. Norman F. Ramsey, Jr., Columbia University;
Professor Robert F. Rinehart, Ashland College, Ashland, Ohio;
Sister Rose M. Cook, Loretto Heights College, Loretto, Colorado;
Professor Gabriel Szegö, Washington University;
Mr. Charles Brown Tompkins, Ann Arbor, Michigan;
Professor Marshal Henry Tyler, Rhode Island State College;
Dr. Yun Chiao Yeh, Shanghai, China.

As nominee of the Mathematical Association of America: Professor B. F. Finkel.

As nominee of Yale University: Dr. Max Zorn, Yale University.

As nominee of Wellesley College: Miss Melita A. Holly, Thurston School, Pittsburgh, Pennsylvania.

The ordinary membership in the Society is now 1782, including 49 nominees of institutional members and 81 life members. There are also 24 institutional members. The total attendance of members at all meetings in 1934 was 916; the number of papers read was 343; the number of members attending at least one meeting was 619.

The reports of the Treasurer and of the auditors (Professors B. P. Gill and H. W. Reddick, and Mr. J. J. Tanzola) showed a balance of $4763.01, exclusive of the balances in the Bulletin, Transactions, Colloquium, Journal, Library, Sinking Fund, and special funds. The Society's Endowment Fund, invested in securities of value $72,568.26 (as amortized March, 1934), yielded in 1934 a net income of $2955.33; institutional memberships for the year amounted to $1365. The amount received from sales of the Society's publications was $9022.49; this does not include receipts from Bliss's *Algebraic Functions*, which have been transmitted to the Revolving Book Fund of the National Research Council. During the year special contributions were received to the amount of $1020.64. The trustees adopted a budget for 1935 showing estimated expenditures and receipts as $41,926.25 and $47,495.05, respectively. The Librarian reported that the Library of the Society now contains 8431 volumes.

Associate Secretary Ingraham reported that encouraging progress had been made up to date in the campaign to stabilize the Society's finances.

The following appointments were announced: as representative on the National Research Council to succeed Professor A. B. Coble, Professor J. H. Van Vleck; as representatives on the

Council of the American Association for the Advancement of Science for 1935, Professors M. H. Ingraham and C. N. Moore; as nominating committee, Professors J. L. Walsh (chairman), E. T. Bell, L. P. Eisenhart, A. J. Kempner, and E. P. Lane; as committee on arrangements for the summer meeting at Ann Arbor, Professors T. H. Hildebrandt (chairman), W. B. Ford, V. G. Grove, M. H. Ingraham, and G. Y. Rainich; as committee on arrangements for the annual meeting at St. Louis, Professors W. H. Roever (chairman), J. E. Case, M. H. Ingraham, and P. R. Rider, and Dr. W. O. Pennell; as committee on Josiah Willard Gibbs Lecturer for 1935 and 1936, Professors G. C. Evans (chairman), H. W. March, and J. D. Tamarkin; as committee on arrangements for the Thanksgiving meeting at the University of Kentucky, Professors P. P. Boyd (chairman), L. W. Cohen, H. H. Downing, C. G. Latimer, F. E. LeStourgeon.

It was reported that the Annual Meeting of the Society for 1935 would be held in St. Louis in connection with the meetings of the American Association for the Advancement of Science and that a meeting of the Society would be held at the University of Kentucky during the Thanksgiving recess next fall. The Council accepted the kind invitation of Harvard University to hold the Summer Meeting in 1936 at that institution. This will be the occasion of the tercentenary celebration of the founding of Harvard University. An invitation from George Washington University to hold the Annual Meeting at Washington in 1936 in connection with the meetings of the American Association for the Advancement of Science was received with thanks to the inviting institution.

It was reported that Professor E. W. Chittenden had accepted the invitation of the Council to give the Cambridge Colloquium in 1936. Upon recommendation of the Committee on the Visiting Lectureship of the Society, the Council voted to appoint Mr. T. Vijayaraghavan of the University of Dacca, Bengal, India, to this lectureship for 1936.

Through the courtesy of the ladies of the Departments of Mathematics of the University of Pittsburgh and the Carnegie Institute of Technology, tea was served to the visiting mathematicians and their guests on Thursday afternoon in the Faculty Room of the Carnegie Institute of Technology. On New Year's Eve the group was the guest of the local mathematicians at a delightful informal party.

The attendance included the following two hundred sixty-two members:

C. R. Adams, R. P. Agnew, E. S. Akeley, O. P. Akers, N. L. Anderson, R. C. Archibald, B. M. Armstrong, C. S. Atchison, W. L. Ayres, R. W. Babcock, R. P. Bailey, S. F. Barber, I. A. Barnett, S. R. Benedict, A. A. Bennett, William Betz, H. R. Beveridge, G. D. Birkhoff, A. H. Black, H. L. Black, Henry Blumberg, Salomon Bochner, Paul Boeder, J. W. Bower, M. G. Boyce, Richard Brauer, Gregory Breit, W. C. Brenke, R. W. Brink, H. W. Brinkmann, A. B. Brown, G. S. Bruton, H. E. Buchanan, C. T. Bumer, R. S. Burington, Keivin Burns, W. E. Byrne, S. S. Cairns, W. D. Cairns, Helen Calkins, R. H. Cameron, I. S. Carroll, G. G. Chambers, L. H. Chambers, R. V. Churchill, E. H. Clarke, W. S. Claytor, A. B. Coble, L. W. Cohen, R. C. Colwell, E. G. H. Comfort, T. F. Cope, A. H. Copeland, L. P. Copeland, Richard Courant, A. T. Craig, C. M. Cramlet, M. M. Culver, C. H. Currier, H. B. Curry, D. R. Curtiss, J. H. Curtiss, E. H. Cutler, H. T. Davis, L. A. V. DeCleene, L. L. Dines, P. S. Donchian, J. L. Doob, Arnold Dresden, J. C. Durand, W. H. Durfee, P. S. Dwyer, M. C. Eide, G. W. Evans, H. P. Evans, H. S. Everett, B. F. Finkel, C. H. Fischer, M. M. Flood, F. A. Foraker, J. S. Frame, A. H. Frink, Orrin Frink, J. J. Gergen, F. J. Gerst, D. C. Gillespie, J. B. Glasgow, Michael Goldberg, F. L. Griffin, L. W. Griffiths, V. G. Grove, B. L. Hagen, E. H. Hanson, W. L. Hart, M. L. Hartung, J. O. Hassler, Alan Hazeltine, E. R. Hedrick, D. M. Hickey, H. C. Hicks, T. H. Hildebrandt, D. L. Holl, T. R. Hollcroft, C. M. Huber, E. M. Hull, Ralph Hull, W. A. Hurwitz, C. A. Hutchinson, M. H. Ingraham, Dunham Jackson, R. L. Jeffery, R. P. Johnson, H. S. Kaltenborn, E. R. van Kampen, D. E. Kearney, H. J. Kersten, S. H. Kimball, J. R. Kline, P. A. Knedler, E. E. Knight, J. C. Knipp, W. D. Lambert, K. W. Lamson, R. E. Langer, G. A. Larew, C. G. Latimer, V. V. Latshaw, Solomon Lefschetz, D. D. Leib, Jack Levine, D. C. Lewis, F. P. Lewis, Marie Litzinger, Dorothy McCoy, N. H. McCoy, J. V. McKelvey, E. J. McShane, L. A. MacColl, C. C. MacDuffee, Saunders MacLane, H. F. MacNeish, Morris Marden, A. E. Meder, D. C. Miller, W. I. Miller, U. G. Mitchell, E. C. Molina, Deane Montgomery, C. N. Moore, T. W. Moore, J. C. Morehead, Max Morris, Richard Morris, W. L. Morris, Marston Morse, Z. I. Mosesson, David Moskovitz, E. J. Moulton, F. D. Murnaghan, J. R. Musselman, S. B. Myers, J. J. Nassau, D. S. Nathan, J. H. Neelley, A. L. Nelson, M. M. Ness, C. O. Oakley, Alta Odoms, F. C. Ogg, Rufus Oldenburger, E. G. Olds, Oystein Ore, E. R. Ott, F. W. Owens, H. B. Owens, F. W. Perkins, H. B. Phillips, H. H. Pixley, Hillel Poritsky, G. B. Price, H. A. Rademacher, Tibor Radó, G. Y. Rainich, S. M. Rambo, W. C. Randels, L. J. Reed, M. S. Rees, W. D. Reeve, C. N. Reynolds, C. E. Rhodes, D. E. Richmond, H. L. Rietz, N. C. Riggs, H. P. Robertson, Robin Robinson, W. H. Roever, C. F. Roos, J. B. Rosenbach, W. E. Roth, N. E. Rutt, E. A. Saibel, George Sauté, I. J. Schoenberg, H. C. Shaub, C. G. Shover, W. G. Simon, M. E. Sinclair, E. B. Skinner, Joseph Slepian, C. E. Smith, T. L. Smith, W. M. Smith, A. E. Staniland, G. W. Starcher, H. E. Stelson, E. B. Stouffer, W. F. G. Swann, J. L. Synge, Gabriel Szegö, J. D. Tamarkin, J. H. Taylor, J. S. Taylor, M. E. Taylor, M. M. Taylor, C. F. Thomas, J. M. Thomas, Stepan Timoshenko, E. W. Titt, C. C. Torrance, J. I. Tracey, W. J. Trjitzinsky, A. W. Tucker, B. M. Turner, F. E. Ulrich, H. S.

Vandiver, J. H. Van Vleck, G. W. Walker, R. J. Walker, J. L. Walsh, Morgan Ward, J. H. Weaver, D. W. Weeks, F. M. Weida, M. J. Weiss, C. H. Wheeler, G. T. Whyburn, D. V. Widder, Norbert Wiener, E. P. Wigner, R. L. Wilder, S. S. Wilks, K. P. Williams, C. O. Williamson, E. B. Wilson, E. W. Wilson, Wilfrid Wilson, Aurel Wintner, E. E. Witmer, E. W. Woolard, F. L. Wren, M. M. Young, Max Zorn.

Titles and cross references to abstracts of papers read at the regular sessions follow below. Mr. Laporte was introduced by Professor Rainich, Dr. Kanarik by Professor Tamarkin, Mr. Cannon by Professor Murnaghan, Dr. Knipp by Professor Foraker, Mr. Sewell by Professor Walsh, Mr. Erskine by Professor Murnaghan, Mr. Heins by Professor Franklin, Dr. Robertson by Professor Tamarkin, Dr. Taussky and Professor Rademacher by Professor Kline, and Mr. Petrie by Professor Olds.

The papers were read as follows: papers numbered 1 to 9 in the General Session on Thursday morning, Professor A. B. Coble presiding; papers numbered 10 to 27 in the Section for Algebra on Thursday afternoon, Professor Oystein Ore presiding; papers numbered 28 to 37 in the Section for Analysis on Thursday afternoon, Professor Marston Morse presiding; papers numbered 38 to 45 in the Session on Mathematical Statistics and Applied Mathematics on Thursday afternoon, Professors H. L. Rietz and C. S. Atchison presiding; papers numbered 46 to 67 at the Session for Analysis on Friday morning, Professor J. D. Tamarkin presiding; papers numbered 68 to 75 at the Session on Algebra on Friday morning, Professor C. C. MacDuffee presiding; papers numbered 76 to 87 at the General Session on Saturday morning, Professor Norbert Wiener presiding. Professor Coble presided at the Annual Business Meeting on Friday afternoon and Professor Lefschetz at the General Meeting held immediately thereafter and also at the Willard Gibbs Lecture. Professor H. P. Robertson presided at the Joint Session with the American Physical Society on Saturday afternoon and Professor H. L. Rietz at the Joint Session on Monday morning. The papers whose abstract numbers are followed by the letter *t* were read by title.

1. *On the evaluation of the characteristic numbers of the Laplacean difference equation in rectangular regions*, by Professor H. C. Hicks. (Abstract No. 41-1-5.)

2. *Pseudominimal hypersurfaces in euclidean four-space*, by

Mr. Otto Laporte and Professor G. Y. Rainich. (Abstract No. 41-1-6.)

3. *A set of completely independent postulates for differentiation*, by Professor F. W. Perkins. (Abstract No. 41-1-7.)

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36. *Non-separable metric spaces*, by Dr. Deane Montgomery (National Research Fellow). (Abstract No. 41-1-31.)

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41. *A sequence exhibiting correlation due to common elements*, by Dr. C. H. Fischer. (Abstract No. 41-1-34.)

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52. *Concerning the connectivity of limiting sets*, by Professor G. T. Whyburn. (Abstract No. 41-1-42-*t*.)

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61. *The radius of univalency of analytic functions*, by Dr. M. S. Robertson (National Research Fellow). (Abstract No. 41-1-49-*t*.)

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64. *Differential invariant theory of alternating tensors*, by Dr. C. M. Cramlet. (Abstract No. 41-1-64.)

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66. *On complete topological spaces*, by Professor John von Neumann. (Abstract No. 41-1-67-*t*.)

67. *Almost periodic functions in groups*. II, by Professors Salomon Bochner and John von Neumann. (Abstract No. 41-1-68-*t*.)

68. *Equivalence of multilinear forms singular on one index*, by Dr. Rufus Oldenburger. (Abstract No. 41-1-51.)

69. *On the rational canonical form of a function of a matrix*, by Professor N. H. McCoy. (Abstract No. 40-11-347.)

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71. *On the irreducible representations of hyperorthogonal groups* (preliminary report), by Dr. J. S. Frame. (Abstract No. 41-1-52.)

72. *Fields as class fields of different reference fields*, by Dr. Olga Taussky. (Abstract No. 41-1-53.)

73. *Some remarks in the theory of algebraic functions*, by Dr. Max Zorn. (Abstract No. 41-1-69.)

74. *Hypergroups* (preliminary report), by Professor H. S. Wall. (Abstract No. 41-1-70-*t*.)

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81. *A classification of generating functions*, by Professor D. V. Widder. (Abstract No. 41-1-58.)

82. *On normal division algebras of prime degree* (preliminary report), by Professor Richard Brauer. (Abstract No. 41-1-59.)

83. *A characterization of manifold boundaries in E_n dependent only on lower dimensional connectivities of the complement*, by Professor R. L. Wilder. (Abstract No. 40-11-354.)

84. *A problem concerning orthogonal polynomials*, by Professor Gabriel Szegö. (Abstract No. 41-1-72-*t*.)

85. *A representation of the most general positive harmonic function in a bounded domain in three-dimensional euclidean space whose boundary points are conical*, by Professor A. J. Maria. (Abstract No. 41-1-73-*t*.)

86. *On the behavior of a harmonic function in the neighborhood of the boundary*, by Professor A. J. Maria. (Abstract No. 41-1-74-*t*.)

87. *On the problem of Pfaff*, by Professor C. M. Cramlet. (Abstract No. 41-1-63-*t*.)

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HILBERT-BERNAYS ON PROOF-THEORY

Grundlagen der Mathematik, Volume I. By D. Hilbert and P. Bernays. (Grundlehren der Mathematischen Wissenschaften, Volume XL.) Berlin, Springer, 1934. xii+471 pp.

This is undoubtedly the most important book on the foundations of mathematics since Whitehead and Russell's *Principia Mathematica*, for it offers an authoritative formulation of the famous Hilbert *Proof-theory*. All the recent work in the foundations has been dominated by the discovery of contradictions in the body of mathematics, especially in Cantor's *Mengenlehre*. There have been many attempts to avoid this difficulty by using axiom systems so limited that the known contradictions could not arise. Hilbert, however, planned a direct attack on the difficulty: an attempt to prove that, in a suitably limited system, no *new* contradiction could ever arise. Bernays here discusses the cases in which Hilbert's plan has succeeded. In brief, the Hilbert school has developed a powerful and fascinating method for investigating mathematical proofs and has shown by these methods that a large part of elementary number theory is *consistent* (free from contradiction). However, the extension to more complicated branches of mathematics has met with serious obstacles.

How is it possible to show that a mathematical system is consistent? Only by means of a thoroughgoing formalization of the axioms and proofs of that system. In other words, the logical methods usually used uncritically in carrying out a mathematical proof must themselves be subjected to mathematical formulation. This is possible by means of the calculus of propositions, which was developed by Peano and by Russell and Whitehead. In this calculus, all the axioms of logic and mathematics can be precisely and symbolically expressed. Furthermore, the operations of logic are all reduced to a few simple, mechanical rules of procedure. A proof must thus start with one or more known axioms, and must proceed step by step, each step following some one of the mechanical rules. Any formal proof is thus finite and combinatorial in character, and hence the possibility that some proof might lead to a contradiction can be investigated by combinatorial methods. This is Hilbert's plan of attack.

But this finite analysis of formal proofs must itself be mathematical and so must itself involve proofs. These latter proofs belong to metamathematics—they are not the mathematics to be investigated; they are rather the tools of the investigation. For example, any general study of proofs will need some sort of complete induction on the number of steps in a proof. This process of induction, together with the other tools needed in the investigation, is essentially *finite* in character. Bernays has explained excellently exactly wherein this finiteness consists. Roughly speaking, finite arguments about numbers are those which can be grasped perceptually (that is, which are *anschaulich überblickbar*). In particular, the existence of a number with some property has a finite meaning only when there is a definite method whereby some such number can be constructed. In this respect, finite theorems are subject to the intuitionistic logic of Brouwer. This means that the tools of proof-theory are to be finite methods which are themselves clearly consistent.

Symbolic logic, the next prerequisite for proof-theory, is developed in §§3–5 in a masterly fashion. In the calculus of propositions the usual operators —"and," "or," "implies," "not," and "equivalent"—are introduced, both by means of axioms and by the superior method of truth-value tables. (For example, "not" is the operator which produces a false proposition from a true one.) For the calculus of propositional functions the author uses a symmetric system of axioms and rules concerning the two operators "for all x" and "there exists an x". The rules of procedure in this system include the usual rule of inference, which allows us to assert the conclusion of a true theorem "A implies B" once the hypothesis A has been established, and the rule of substitution, according to which variables may be replaced by constants or combinations of variables. An additional rule allows us to *rename* the *apparent* variable x in "for all x" or "there exists an x." A final section discusses a method of adjoining to this calculus the relation of equality.

The sixth section of the book introduces the first typical application of proof-theory to an infinite system: the demonstration that a certain axiom system for whole numbers is consistent. This system consists of the calculus of propositional functions, including equality, and the Peano axioms for whole numbers, excluding the principle of mathematical induction. The Peano axioms are slightly modified, in that the function $a'=a+1$ is used instead of the relation "b is the successor of a". If this system of axioms were inconsistent, then there would be a proof starting from these axioms and leading to a contradiction of the form $0\neq 0$. In the simplest case, this proof would involve no apparent variables. In such a proof, all free variables can be eliminated, for any application of the rule of substitution can be moved back in the proof until the substitution in question takes place right in the original axioms. All the formulas of the proof are then simple *numerical* formulas—equalities and inequalities between concrete numbers, together with combinations of such by the operations of the propositional calculus. Any such numerical formula is either "true" or "false" in a finite, constructive sense. Furthermore, the particular axioms from which the proof starts are all "true" in this sense, while all the rules of procedure give true results when applied to true premises. In particular, if two formulas "S" and "S implies T" are true, then the conclusion "T" of the inference is likewise true. Hence all the formulas derived in the course of the proof are true, so that a contradiction $0\neq 0$ at the end of the proof cannot occur.

This result must next be extended to proofs which do involve apparent variables. This is possible with the methods of Herbrand and Presburger. The formulas of such a proof are not all numerical formulas, but it is possible to associate with any one of the formulas several corresponding *reduced* numerical formulas. For example, $(Ex)\ (a<x\ \&\ x<b)$ would have the reduced form $0<b\ \&\ (a+1)<b$. Here, as in the other cases, the reduced formula is effectively equivalent to the original. Hence any formula is called *verifiable* if one of its reduced forms is true. As before, all the formulas of a proof are verifiable, so that no contradiction can arise.

Much of the rest of the book is concerned with extensions of this method of establishing consistency by means of "verifiable" formulas and methods of "reduction." In particular, these ideas are still applicable if the axiom of mathematical induction is added to the other Peano axioms. In other words,

the combination of the Peano axioms for whole numbers, the axioms for equality, and the calculus of propositional functions gives a consistent system.

This, however, does not prove that all ordinary number theory is consistent, for the usual development of whole numbers uses not only these axioms, but also certain *recursive definitions* for functions such as $a+b$. Such recursive definitions, unlike the ordinary explicit definitions, can at times lead to contradictions in systems which would otherwise be consistent. Hence the next task is an investigation of the consistency of recursive definitions. The author here considers any system of verifiable axioms for number theory, together with mathematical induction, the axioms for equality, and the calculus of propositions for *free* variables only. The addition of recursive definitions to such a system will not give rise to any contradiction, as is shown by a modification of the previous methods. The exclusion of apparent variables is essential to this result, but even with this limitation much can be developed. This the author shows by developing the usual theorems for prime decomposition by means of such recursions.

The last major topic is the analysis of descriptive functions, of the type "the so-and-so" or "the smallest number with such and such a property". This study was begun by Whitehead and Russell. A considerable extension of their method enables Hilbert and Bernays to show that such descriptive functions can always be eliminated: a theorem which does not involve descriptive functions, but which has been proved by the use of descriptive functions, can also be proved without descriptive functions. This result has many applications, in particular to a study of recursive definitions.

This book also contains a wealth of other significant results: a system of positive logic (§3); the deduction theorem (§4); the contributions of Skolem, Löwenheim, Behmann, and others to the *Entscheidungsproblem* (§§4–5); the proofs of several additional rules of procedure (§4); the Ackermann-Péter "folded" recursions which cannot be reduced to primitive recursions (§7); a discussion of normal forms in the propositional calculus with equality (§5); a critique of axioms for equality (§7); and several interesting axiom-systems for parts of number-theory. Indeed, if this book has a fault, it is only in that Professor Bernays' encyclopedic knowledge has led him to include so much that the main theme becomes at times obscured. Nevertheless, the book is very carefully and clearly written. Many of the arguments are of necessity formal in character, but if at such points the reader will bear in mind the non-formal interpretation of the discussion, he will have little trouble. The author has succeeded unusually well in explaining an abstract subject without assuming any previous special knowledge on the part of the reader.

The chief open question is that of extending the results. This volume has established the consistency of a system of recursive number theory; but for more extensive systems, the chief result is the preliminary step of eliminating the descriptive functions. To include all of number theory in a consistency proof, fundamental changes in the method are necessary, as the authors recognize. In spite of the optimism which Hilbert expresses in his preface, the well known results of Gödel point to almost insurmountable difficulties in the program of proof-theory. The second volume of this work, with its discussion of

these questions, will be awaited with great interest. It may be that still other points of view will be necessary to complete the foundations of mathematics so well begun by the Hilbert proof-theory.

SAUNDERS MACLANE

SZEGÖ ON JACOBI POLYNOMIALS

Asymptotische Entwicklungen der Jacobischen Polynome. By Gabriel Szegö. Schriften der Königsberger Gelehrten Gesellschaft. Jahr 10, Naturwissenschaftliche Klasse, Heft 3, 1933, pp. 35–111 (1–77).

One of the interesting features in the development of analysis in the twentieth century is the remarkable growth, in various directions, of the theory of orthogonal functions. Two brilliant achievements on the threshold of this century—Fejér's paper on Fourier series and Fredholm's papers on integral equations—have been acting as a powerful inspiring source of attraction, inviting analysts to delve deeper into the theory of orthogonal functions and their applications. First come, due to their simplicity, the trigonometric functions $\{\sin mx, \cos mx\}$ which serve as a yardstick for orthogonal functions in general. Next we may consider orthogonal polynomials, of which Jacobi polynomials are a special case.

Let us recall the general definition of orthogonal polynomials. A weight-function $p(x)$, non-negative in a given interval (a, b), finite or infinite, and such that all "moments" $\int_a^b p(x)x^r dx = \alpha_r$ exist, $(r=0, 1, 2, \cdots)$, with $\alpha_0 > 0$, gives rise to a unique system of orthogonal and normal polynomials $\phi_n(x) = a_n x^n + \cdots$, $(n=0, 1, \cdots; a_n > 0)$, so that

$$\int_a^b p(x)\phi_m(x)\phi_n(x)dx = 0, \ (m \neq n), \quad = 1, \ (m = n), \quad (m, n = 0, 1, \cdots). \tag{1}$$

On the basis of (1), we obtain the following expansion of an "arbitrary" function:

$$f(x) \sim \sum_{n=0}^{\infty} f_n \phi_n(x), \quad \text{with} \quad f_n = \int_a^b p(x)f(x)\phi_n(x)dx, \tag{2}$$

and this constitutes the most interesting and important application of the polynomials $\phi_n(x)$ in analysis, as well as in mathematical physics, mathematical statistics, etc.

The oldest and best known are Legendre polynomials, derived from (1) with (a, b) finite, say $(-1, 1)$, and $p(x) \equiv 1$. Their direct generalization are Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$: $(a, b) = (-1, 1)$, $p(x) = (1-x)^\alpha(1+x)^\beta$, $\alpha, \beta > -1$. In case of an infinite interval, the most important are the polynomials of Laguerre: $(a, b) = (0, \infty)$, $p(x) = x^\alpha e^{-x}$, $\alpha > -1$, and those of Hermite: $(a, b) = (-\infty, \infty)$, $p(x) = e^{-x^2}$. These four kinds of orthogonal polynomials constitute what may be considered as one single family of "classical" polynomials, where Jacobi polynomials, from many points of view, represent the most typical member. In fact, by assigning to α, β certain finite or limiting values, we get Legendre polynomials $(\alpha = \beta = 0)$, trigonometric polynomials $(\alpha = \beta = -1/2)$,

also the polynomials of Laguerre and Hermite. A study of Jacobi polynomials is therefore of special value and interest.

In the book under review Professor Szegö, whom we are now happy to greet in our midst, takes up the *asymptotic properties of* $P_n^{(\alpha,\beta)}(x)$, that is, their behavior when the degree n increases indefinitely. A glance at the expansion (2) shows at once the importance of these properties, for they influence to the greatest extent the convergence (or summability) of (2); they also play a predominant part in many other problems, like mechanical quadratures. To this study Professor Szegö brings his well known mastery of the finest tools of analysis, and his lucidity and elegance of exposition, which are worthy acquisitions from his eminent teacher Fejér.

The asymptotic expression for $P_n^{(\alpha,\beta)}(x)$, with arbitrary α, $\beta(>-1)$, was derived by Darboux, that for Legendre polynomials by Laplace. Trigonometric functions are exclusively employed in these expressions. While this makes for simplicity, we also find great inconvenience, due to the fact that the expressions in question hold *inside* the interval of orthogonality only. Here Szegö utilizes *Bessel functions*, instead of trigonometric ones,* and in this way he is able to derive asymptotic expressions *uniformly valid in any left-hand neighborhood of the end point* $x=1$. This is the main object of the book, the importance of which is evident to any worker in this field. It suffices to recall the laborious considerations and computations necessitated in the past by the lack of such asymptotic expressions, when evaluating for Legendre polynomials, for example, Lebesgue constants corresponding to the end points $x=\pm 1$. Szegö himself shows the application of his asymptotic expressions to various interesting problems involving Jacobi polynomials. Their solution now becomes strikingly simple.

The book consists of four chapters preceded by an Introduction, where one finds an outline of the general purpose of this work and of the methods employed, and is closed by an Appendix dealing with the so-called "Associated Functions". The methods employed and the results obtained are an extension of those to be found in a previous paper by the author on Legendre polynomials.

The first chapter is devoted to marshalling necessary data and formulas, and to formulating the fundamental results obtained. We first find here the definition, through the generating function, of $P_n^{(\alpha,\beta)}(x)$ for any real α, β (the book deals chiefly with the *orthogonal* Jacobi polynomials, that is, α, $\beta>-1$), with emphasis upon the symmetric case of "ultraspherical" polynomials $P_n^{(\mu)}(x)$ $(=\text{const. } xP_n^{(\alpha,\alpha)}(x)$, $\mu=\alpha+1/2)$. This definition yields at once the explicit expression of $P_n^{(\alpha,\beta)}(x)$ and some of their fundamental properties:

$$P_n^{(\alpha,\beta)}(x) \equiv (-1)^n P_n^{(\beta,\alpha)}(-x); \quad \frac{d}{dx}P_n^{(\alpha,\beta)}(x) = \frac{n+\alpha+\beta+1}{2}P_{n-1}^{(\alpha+1,\beta+1)}(x).$$

The first property reduces the discussion of $P_n^{(\alpha,\beta)}(x)$ in $(-1, 1)$, which is the interval of orthogonality, if α, $\beta>-1$, to that in $(0, 1)$; the second property is a very important one; it shows, for example, that the asymptotic expression for $P_n^{(\alpha,\beta)}(x)$ can be differentiated and thus throws more light on the behavior

* For Legendre polynomials, this was partly done previously by Watson and Hilb.

of the remainder. The author next gives some properties of Bessel functions $J_\nu(\xi)$ of the first kind, emphasizing, for future needs, their asymptotic behavior for $\xi \to +0$ and $\xi \to +\infty$, also $\max \xi^{1/2}|J_\nu(\xi)|$ for $\xi \geqq 0$. He further introduces, as a generalization of $J_\nu(\xi)$, certain functions $J_{\nu,m}(\xi)$ and, by expressing these in terms of $J_\nu(\xi)$, he shows how to estimate their order of magnitude for $\xi \to 0$, $\xi \to \infty$. The chapter closes by stating the fundamental theorems of this book giving the asymptotic expressions of $P_n^{(\mu)}(\cos\theta)$ (Theorem 1) and of $P_n^{(\alpha,\beta)}(\cos\theta)$ (Theorem 2). Thus

$$(3)\quad P_n^{(\mu)}(\cos\theta) \sim \sum_{m=0}^{\infty} f_m(\theta)\frac{\theta^{m-\mu+1/2}}{(n+\mu)^{-m\mu+1/2}} J_{\mu-m-1/2}[(n+\mu)\theta], \qquad (0<\theta<\pi).$$

Here the essentially novel and important feature, as pointed out above, *is the use of Bessel functions*. The $f_m(\theta)$ are "elementary" functions, regular for $0 \leqq \theta < \pi$; the first few are given explicitly. The author also gives, for $n \to \infty$, an o-estimate of the remainder in the above expressions, broken up after a certain number of terms, for $\theta \geqq c/n$ and $\theta \leqq c/n$, (c fixed, >0). We also find here an asymptotic expansion of $P_n^{(\mu)}(\cos\theta)$ in terms of the orthodox trigonometric functions. The wider power of the asymptotic expansions employing Bessel functions is illustrated by the fact that the principal term alone suffices to improve Darboux's classical formula for $P_n^{(\alpha,\beta)}(x)$. Various possible applications of the foregoing expansions are indicated, one of the most important of which deals with mechanical quadratures (Theorem 3, see below).

Chapter 2 is devoted to the proof of Theorem 1, the asymptotic expansions of $P_n^{(\mu)}(\cos\theta)$. The main point is the expression of $P_n^{(\mu)}(x)$ as a contour integral (making use of the generating function). By choosing properly the contour of integration, so that the integrals involved become identical with those figuring in the definition of $J_\nu(\xi)$, we obtain (3). Another choice of the contour yields the orthodox (trigonometric) expansion for $P_n^{(\mu)}(\cos\theta)$.

Chapter 3 in its first half deals with Theorem 2, the asymptotic expansion of $P_n^{(\alpha,\beta)}(\cos\theta)$ (for any real α, β; $0 < \theta \leqq \pi - \epsilon$; $\epsilon > 0$, arbitrarily fixed, $0 < \epsilon < 1$). The analysis is again based on the integral representation of $P_n^{(\alpha,\beta)}(x)$ and requires here a more penetrating investigation of the generating function and the introduction of the generalized functions $J_{\nu,m}(\xi)$. The author shows that the remainder for $\theta \leqq c/n$ can be estimated in another way, by writing

$$n^{-\alpha}P_n^{(\alpha,\beta)}\left(\cos\frac{u}{n}\right) \sim \phi_0(u) + \frac{\phi_1(u)}{n} + \frac{\phi_2(u)}{n^2} + \cdots,$$

(where $\phi_m(u)$ are integral functions of u), and making use of the estimate already arrived at for $\theta \geqq c/n$. He further points out that for integral values of α there exists a certain interval for θ where the above expansion converges in the ordinary sense (a similar remark applies to the expansion (3)). Finally, a more precise o-estimate of the remainder is obtained, for some special values of α, by o-estimating the mth term of the expansion in question. The second half of Chapter 3 gives various interesting applications of the foregoing formulas. We thus find, for example,

$$(4)\quad \begin{aligned} &|P_n^{(\alpha,\beta)}(x)| \leqq A_n^{-1/2}(1-x)^{-\alpha/2-1/4}, && (\alpha \geqq -1/2),\\ &|P_n^{(\alpha,\beta)}(x)| \leqq A_n^{\max(\alpha,-1/2)}, && -1+\epsilon \leqq x \leqq 1), \end{aligned}$$

where A is an appropriate constant, independent of n. We are also enabled to treat in a very simple manner the asymptotic behavior $(n\to\infty)$ of

$$\max\,(1-x)^{\lambda}(1+x)^{\mu}\left|P_n^{(\alpha,\beta)}(x)\right|,\quad(-1\leqq x\leqq 1),\ (\lambda,\mu\text{ fixed},\geqq 0),\tag{5}$$

which, for $\lambda=\alpha/2+1/4$, $\mu=\beta/2+1/4$, was treated by S. Bernstein by means of the differential equation for Jacobi polynomials. With $\alpha=\beta=0$, $\lambda=\mu=1/4$ or $\alpha=\beta=1$, $\lambda=\mu=3/4$ we obtain, respectively, the boundedness in $(-1, 1)$ of $n^{1/2}(1-x^2)^{1/4}\left|P_n(x)\right|$ (Stieltjes), $n^{-1/2}(1-x^2)^{3/4}\left|P_n'(x)\right|$ (Kogbetliantz) (P_n = Legendre polynomial). Such estimates as (4) or (5) are very important when we are discussing, for example, the convergence of (2). In a similar manner, we can readily estimate asymptotically $(n\to\infty)$ many definite integrals involving $P_n^{(\alpha,\beta)}(x)$, such as

$$\int_{-1}^{1}\left|\frac{d}{dx}P_n^{(\alpha,\beta)}(x)\right|dx,\quad\int_{-1}^{1}(1-x)^{\lambda}(1+x)^{\mu}\left|P_n^{(\alpha,\beta)}(x)\right|,\quad(\lambda,\mu\text{ fixed},>-1).$$

Finally comes the most important application, the asymptotic estimate of the so-called Lebesgue constants

$$\rho_n^{(\alpha,\beta)}(x)=\int_{-1}^{1}(1-t)^{\alpha}(1+t)^{\beta}\left|\sum_{m=0}^{n}\phi_m(x)\phi_m(t)\right|dt,\tag{6}$$

where $\phi_m(x)$ are *normalized* Jacobi polynomials, *at the end point* $x=1$. These constants, as is well known, are of paramount importance in the discussion of the convergence properties of (2). The estimate of $P_n^{(\alpha,\beta)}(1)$ now follows very readily from the above asymptotic expressions, due to the fact that they hold uniformly in any left-hand neighborhood of $x=1$ (in contrast with the classical ones). The climax is reached in the closing section where the author proves the "*equiconvergence theorem*". Let the L-measurable function $f(x)$ be such that

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\left|f(x)\right|dx,\ \int_{-1}^{1}(1-x)^{\alpha/2-1/4}(1+x)^{\beta/2-1/4}\left|f(x)\right|dx,\ (\alpha,\beta>-1),$$

exist. Denote by $s_n(x)$ the nth partial sum of the expansion (2) of $f(x)$ in series of Jacobi polynomials, by $\sigma_n(x)\equiv\sigma_n(\cos\theta)$, the nth partial sum of the Fourier cosine series expansion of $F(\theta)=(1-\cos\theta)^{\alpha}(1+\cos\theta)^{\beta}f(\cos\theta)\left|\sin\theta\right|$. Then

$$\lim_{n\to\infty}\left[s_n(x)-(1-x)^{-\alpha-1/2}(1+x)^{-\beta-1/2}\sigma_n(x)\right]=0,\quad(-1<x<1),$$

uniformly for $-1+\epsilon\leqq x\leqq 1-\epsilon$. Moreover, if $f(x)$ is bounded in $(-1, 1)$, then we can compare (2) with the ordinary Fourier series expansion of the function $f(\cos\theta)$ itself. *Thus the discussion of the convergence (or summability) of* (2) *is reduced to that of an ordinary Fourier series*, which is the goal (of course, not always attainable) for all expansions in series of orthogonal functions.

Chapter 4 is devoted to the Newton-Cotes mechanical quadrature formula based on the Lagrange interpolation formula *with the zeros of a Jacobi polynomial as abscissas*:

$$\int_{-1}^{1}f(x)dx=\sum_{i=0}^{n}\lambda_i f(x_i)+R_n\qquad\lambda_i=\int_{-1}^{1}\frac{\phi_n(x)}{(x-x_i)\phi_n'(x_i)}dx,\tag{7}$$

$$\left[\phi_n(x)=\text{normalized }P_n^{(\alpha,\beta)}(x),\ P_n^{(\alpha,\beta)}(x_i)=0\right].$$

The novelty and interest of the formula (7) lies in the fact that *the weight function* $p(x)=(1-x)^{\alpha}(1+x)^{\beta}$ *is here omitted* (in contrast with the classical formulas of mechanical quadratures of Gauss' type). On the basis of recent investigations of Fejér and Pólya, the question as to the general convergence of (7), that is, as to the existence of $\lim_{n\to\infty}R_n=0$ for all R-integrable functions $f(x)$, essentially depends upon the signs of the "Cotes coefficients" λ_i. The author shows here that

$$\text{(8)} \qquad \operatorname{sgn}\lambda_i=\operatorname{sgn}K_n(x_i), \quad K_n(x)=\sum_{m=0}^{n}\phi_m(x)\int_{-1}^{1}\phi_m(t)dt.$$

Making use of the Darboux formula for $\sum_{m=0}^{n}\phi_m(x)\phi_m(t)$ and of some orthogonality properties of $K_n(x)$, Szegö is able to investigate the signs of $\{\lambda_i\}$ for some special α, β. A similar investigation for any α, β (>-1) is achieved through the improved (see above) Darboux asymptotic expression for $P_n^{(\alpha,\beta)}(\cos\theta)$. We thus conclude (Theorem 3, Chapter 1): (a) if $\max(\alpha,\beta)>3/2$, then there exists a function $f(x)$, continuous in $(-1, 1)$, for which (7) diverges; (b) if $\max(\alpha,\beta)<3/2$, then (7) converges for all $f(x)$ R-integrable in $(-1, 1)$. The case (b) also holds for $\alpha=\beta=3/2$.* Special cases of Theorem 3 are due to Fejér. The chapter closes with the case $-1/2\leqq\alpha=\beta\leqq 0$, where it is shown that all λ_i are positive.

The Appendix deals briefly with the application of the methods of Chapter 2 in order to obtain asymptotic expansions of the *associated functions*

$$P_n^{\nu}(\cos\theta)=e^{i\nu\pi}\cdot\frac{\Gamma(\nu+1/2)}{\Gamma(1/2)}(2\sin\theta)^{\nu}\cdot\frac{1}{2\pi i}\int_C(1-2w\cos\theta+w^2)^{-\nu-1/2}w^{n+\nu}dw.$$

Following traditional lines, we must try to find some fault with the book under review. We did find one fault—to be sure, of measure zero. The author in setting up the asymptotic expansions for $P_n^{(\mu)}(\cos\theta)$, $P_n^{(\alpha,\beta)}(\cos\theta)$, uses what he calls "elementary functions" $(f_m(\theta), f_{m,\nu}(\theta))$. This designation seems to the reviewer rather vague and thus unnecessary.

At the end of Chapter 1 the author promises to give, by similar methods, an investigation of the asymptotic nature of the polynomials of Laguerre and Hermite. The realization of this promise will be eagerly awaited, we feel certain, by all those who are interested in analysis in general, and in particular in the vast fruitful field of orthogonal polynomials.

J. SHOHAT

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* Probably also for $\max(\alpha,\beta)=3/2$, $\alpha\neq\beta$ (Szegö).

SHORTER NOTICES

Les Surfaces Algébriques non Rationnelles de Genre Arithmétique et Géométrique Nuls. By L. Godeaux. (Actualités Scientifiques et Industrielles, No. 123.) Paris, Hermann, 1934. 33 pp.

The most important invariant of an algebraic curve is its genus. A necessary and sufficient condition that a non-composite curve shall be rational is that its genus be zero. An algebraic surface has an infinite number of genera, each a direct and natural analog of the genus of a curve. These are the arithmetic genus p_a, the geometric genus p_g, and the series of multigenera P_i. It was shown by Castelnuovo, immediately after Enriques had discovered the invariants P_i, that a necessary and sufficient condition for rationality is $p_a=0$, $P_2=0$. Later, each of these scholars actually constructed irrational surfaces having $p_a=0$, $p_g=0$, $P_2\neq 0$.

The present pamphlet clearly states the problem, shows that these illustrations are birationally distinct, gives the further case found by Campedelli, $P_2=2$, and adds a new one for which $P_2=2$, $P_3=4$. The method of proof is entirely algebraic, using only the well known theorems concerning grade, genus, and laws of combination for linear systems of curves on an algebraic surface. An excellent introduction to this memoir is the paper by Roberta F. Johnson, *Involutions of order two associated with the surfaces of genera* $p_a=p_g=0$, $P_2=1$, $P_3=0$ (American Journal of Mathematics, vol. 56 (1934), pp. 199–213).

VIRGIL SNYDER

Mathematical Problems of Radiative Equilibrium. By Eberhard Hopf. Cambridge Tracts in Mathematics and Mathematical Physics. New York, Macmillan. viii+105 pp.

This is an excellent account of the mathematical developments associated with a class of integral equations in which the kernel is a particular type of function of the modulus of the difference of x and y. The author regards his work as applicable to the mathematical problems of radiative equilibrium which arose from the work of Schuster and Schwarzschild on a purely scattering atmosphere; and became more intricate when the atmosphere was regarded as absorbing and emitting radiation as in the work of Schwarzschild, Humphreys, Gold, and Emden on the earth's atmosphere and in the later work of Eddington, Jeans, Milne, and others on stellar atmospheres and planetary nebulae. It may be suggested, however, that the analysis may also be useful for a continuation of the work of E. Lommel, *Die Photometrie der diffusen Zurückwerfung*, published in 1889 in volume 36 of Wiedemann's Annalen der Physik und Chemie, for in this work Lommel obtains an integral equation of the type under consideration, the particular function being the exponential integral which belongs to the type mentioned in this tract. It is impossible to give a brief summary of the analysis, but it can be said that the work is of a high standard throughout, much use being made of inequalities and convergence theorems. It is a useful contribution to modern astrophysics.

HARRY BATEMAN

Les Rayons α, β, γ des Corps Radioactifs en Relation avec la Structure Nucléaire. By Madame P. Curie. (Actualités Scientifiques et Industrielles, No. 62.) Paris, Hermann, 1933. 40 pp.

In this monograph, which is quite non-mathematical, the famous discoverer of radium, whose recent death was a great loss to science, discusses the recent attempts to obtain information about the nature of the nuclei of atoms from a study of the inhomogeneities present in their radiations. Reference is made to the complexities introduced by the recent conceptions of neutrons and positrons. The experimental evidence seems to support Gamow's hypothesis that the nucleus, like Bohr's atom, possesses energy levels, α-particles playing the role of electrons.

F. D. MURNAGHAN

Vorlesungen über Projektive Geometrie. By C. Juel. Berlin, Springer, 1934. xi+287 pp.

C. Juel, who is professor emeritus at the Technical High School of Kopenhagen, has written a book on projective geometry with special consideration of von Staudt's theory of imaginaries.

He assumes a knowledge of ordinary real projective geometry as for example contained in F. Enriques' *Vorlesungen über projektive Geometrie*. Professor Juel has succeeded in giving as clear a treatment as possible of an otherwise cumbersome and somewhat obsolete method. At the present time one is inclined to make use of the much more effective and elegant analytic-geometric method with its great time-saving factor.

Among the particular features of the book may be mentioned chapters on the plane cubic curve and the quadratic transformation.

ARNOLD EMCH

Theorie der Funktionen Mehrerer Komplexen Veränderlichen. By H. Behnke und P. Thullen. Ergebnisse der Mathematik und ihrer Grenzgebiete, Volume 3. Berlin, J. Springer, 1934. 115 pp.

This little book surveys in a compact and scholarly fashion the present status of some basic questions in the yet immature but rapidly developing subject of functions of several complex variables. The authors acknowledge their initial indebtedness to Osgood's *Lehrbuch der Funktionentheorie*, II, 1: *Grundlagen der allgemeinen Theorie der Funktionen Mehrerer Komplexen Grössen*, 1924, comparison with which gives evidence of the progress achieved during the last decade.

The chapter headings suggest the scope of this work: I. Domains over the extended space. II. Geometrical foundations. III. Representation of regular functions by elementary series. IV. Singular manifolds. V. Distribution of zeros and of nonessential singularities. VI. Theory of domains of regularity and of shells of regularity (Regularitätshüllen). VII. Transformation theory. Much of this work is of necessity occupied with extending familiar concepts in an obvious way either from one variable to n, or from real geometry to complex. The authors contrast two traditional types of n-space, one called by Osgood the "space of analysis," and having n hyperplanes at infinity, and the other the projective space having a single hyperplane at infinity. Instead of

retaining throughout both alternatives, the projective space is here adopted as basis for discussion. Significant early concepts that may prove new to the reader include those of pseudo-convexity, cylinder-domain, and a surface of determination for it, dicylinder, Reinhardt domain (of which hyperspheres and dicylinders are examples), circular domain, Hartogs domain (having regard to a single variable), Cartan domain.

The sixth chapter elaborates the implications of the following striking theorem (page 39): "Every function $f(w, z)$ regular in a proper Reinhardt domain K is developable in a power series convergent throughout K and is furthermore analytically extensible throughout the interior of the minimal complete Reinhardt domain containing K." The final chapter discusses the automorphic mapping of a region particularly with respect to the invariance of the center. Use is made of the metric of Carathéodory in which distance between two points, A and B, is given by the upper bound of

$$\log \frac{|f(A) - f(B)| + |1 - f(A)\overline{f(B)}|}{[(1 - f(A)\overline{f(A)})(1 - f(B)\overline{f(B)})]^{1/2}}$$

with respect to all permissible functions, f.

It is clear that the subject is acquiring a content other than as a simple hybrid product of the theory of one complex variable crossed with the geometry of the real plane. One notes, however, as marking presumably a transient aspect, that the results here given are fairly illustrated by the case of two variables, theorems for higher dimensions being as yet limited to relatively trivial generalizations from the important case of $n = 2$. The postulational approach is avoided in this work so that questions of abstract space and even the tractable case of Hilbert space are not suggested although in so far as the particular value of n remains non-significant some of the theory might be expected to depend upon features available under more general conditions.

In generalizing from one variable it is natural to take up first the case of a single function of n independent complex variables, dealing with series expansions, singularities, regions of convergence, and like questions. Problems relating to uniform mapping and to invariants under analytic transformations, when generalized to n variables, suggest the study of simultaneous systems of n functions of n variables, as emphasized by Forsyth in the earliest book on the subject, *Lectures Introductory to the Theory of Functions of Two Complex Variables*, 1914. As remarked above, the last chapter of the present work gives some recent results in this direction, but not in the vectorial language symmetry would seem to demand. Indeed one may feel that despite some progress toward a geometrical viewpoint, the independent variables have remained a trifle too obstinately independent in the present theory and have not been completely merged into the concept of a point in a homogeneous projective space for which a choice of coordinates is extraneous or adventitious.

But whether or not one may have the impression that this discussion of the subject calls for a more independent abstract treatment and broader vision, the reader will not fail to appreciate the concrete details here arranged in logical order with abundant references to the literature and the inclusion of explanatory and illustrative material where most needed.

A. A. Bennett

La Méthode du Champ Self-Consistent. By L. Brillouin. (Actualités Scientifiques et Industrielles, No. 71.) Paris, Hermann, 1933. 46 pp.

This monograph explains a method of approximating a solution of the many body problem in wave mechanics which was given first by Hartree. The problem is to find the wave function appropriate for a system of electrons under the influence of several fixed nuclei and of one another; the method is to represent the desired wave function as a product of separate wave functions, each of which depends only on one of the electrons. Each of these factor wave functions is found by supposing the corresponding electron to move under the influence of the fixed nuclei and a field obtained by averaging the fields of all the other electrons. An account of the modifications due to the "spin" of the electrons and of Pauli's exclusion principle is given.

The treatment is authoritative, and, despite the little space at his disposal, the author does not shirk the essential and obvious difficulties of the problem.

F. D. MURNAGHAN

Fundamentals of Hydro- and Aeromechanics, based on Lectures by L. Prandtl. By O. G. Tietjens (translated by L. Rosenhead). New York, McGraw-Hill, 1934. xvi+270 pp.

This and a companion volume which is reviewed below form the first of a series of monographs known as Engineering Societies Monographs to be published under the auspices of several of the American Engineering Societies. The purpose of the book is to give an account of the theory of hydrodynamics which is as simple and as closely related to experience as possible. No attempt is made to treat the advanced mathematical theory, for which reference is made to Lamb's classical treatise. Part I, pages 1–65, treats hydrostatics and discusses such matters as the equilibrium of gas-filled balloons, temperature effects, surface tension. Part 2, pages 69–104, describes the Lagrangian and Eulerian methods of treatment and gives a brief and intuitive account of the necessary vector analysis. Part 3, pages 107–265, treats the dynamics of non-viscous fluids ending with a brief chapter descriptive of Stokes' treatment of a viscous fluid and referring to Oseen's improvement. Two-dimensional problems and the theory of vortices are treated quite fully and there is a welcome discussion of the effect of compressibility.

The whole book breathes the spirit of the engineer and numerical calculations are frequently given; whenever a formula yields results not in accord with experimental facts the fallacy in the assumptions made is clearly pointed out. It is our opinion that the book fulfills its purpose admirably and it can be highly recommended as a complement to Lamb's work.

F. D. MURNAGHAN

Applied Hydro- and Aeromechanics, based on Lectures by L. Prandtl. By O. G. Tietjens (translated by J. P. Den Hartog). New York, McGraw-Hill, 1934. xvi+311 pp.

This book together with its companion volume, reviewed above, is one of the new series of Engineering Societies Monographs. After a very brief recapitulation of the fundamental laws and a chapter on the laws of similarity, an ex-

tended discussion of problems in hydraulics (flow in pipes and channels) is given (in which references to quite recent work are made). Then follows a good account of the Prandtl boundary layer theory and a discussion of resistance (Stokes' theory, Karman vortices, etc.). The theory of airfoils is amply treated (it being pleasant to see the credit given to Lanchester's pioneer work). The book closes with a chapter on experimental methods and apparatus and an appendix containing some sixty-eight photographs made by the author of various types of flow.

We strongly recommend this work to any student of practical hydrodynamics. As in the case of the companion volume, one interested mainly in the mathematical theory will have to look elsewhere.

F. D. MURNAGHAN

Les Calculs Formels des Séries de Factorielles. By J. Ser. Paris, Gauthier-Villars, 1933. vii+98 pp.

This book has to do with series, such as

$$f_0 - \sum_{n=1}^{\infty} f_n \frac{x(1-x)\cdots(n-1-x)}{1\cdot 2\cdots n},$$

arising in connection with the table of successive differences of the values of a function $f(x)$ for equally spaced values of x. A large number of transformations of these series, and of expansions of particular functions in the series, are given. The treatment is entirely formal, there being only a few passing remarks about questions of convergence. Nothing in the way of a theory is developed; the contents consist chiefly of a collection of special calculations.

Even when due allowance is made for the restrictions that have been imposed intentionally on the subject matter, the book remains rather unsatisfactory. As there is no index, and as the subdivisions of the chapters have no titles, it is difficult to use the book for reference purposes. In many places, because of the lack of full explanations, the meaning is obscure. There are no references to the literature, and there are many typographical errors.

L. A. MACCOLL

Le Mystère et le Paradoxe du Vol Animal. By Émile Batault. Paris, Gauthier-Villars, 1933. 14+236 pp.

Sur l'Excédent de Puissance des Oiseaux. By A. Magnan and A. Planiol. Paris, Hermann, 1933. 25 pp.

Sur l'Excédent de Puissance des Insectes. By A. Magnan and A. Planiol. Paris, Hermann, 1933. 26 pp.

It must have been from the flight of birds that man first conceived the ambition to fly. And it was natural to believe that the solution of the problem lay in the study and imitation of animal flight. The student of the history of science is familiar with the drawings that Leonardo da Vinci made of wing-like appliances for man's use. The difficulty in the problem was that of power. The perfection of the gasoline motor and the screw propeller gave an entirely new approach to the question. As a result the flight of airplanes and the flight of birds differ in fundamental ways, and in consequence the latter has lost some of its interest. As a natural phenomenon, however, the flight of birds is still

worthy of study, and there are many perplexing questions connected with it. The interest in the problem is not only mechanical, but also physiological.

The writer of the first book was a physician who for years was interested in the problem of animal flight. A mechanical aptitude, shown by various inventions, gave him the qualifications for appreciating both the mechanical and physiological aspect of the question he was studying. The author died before the appearance of the work, and the final proofs of the book were handled by his friend Henri Tripier.

The first book starts with a historical account of the problem, beginning with Leonardo da Vinci, he being the first to have worked in a scientific manner. The author then proceeds to anatomical and physiological considerations. A first question is that of the relative muscular equipment of birds and men. Dr. Batault does not agree with the high estimates that some writers have placed on the power that birds develop. He believes that errors have been made by an inadmissible use of the quantity KSV^2 for calculating the resistance overcome. He believes that the muscles that man uses in walking are relatively as powerful as those a bird uses in flight. But he does find that the muscular systems of creatures that fly differ from those of other animals by the rapidity with which they act.

The author of the first book puts questions in a very definite way. For instance, in Chapter 4 we find the heading, "Que fait l'oiseau dans l'air pour s'y soutenir?" Here certainly is the kernel of the problem. Dr. Batault's view of the question is this. The rapid movement of the wing impounds a certain amount of air, which for a moment becomes a part of the flying mass considered. Then the downward movement of the wing being arrested, the air is expelled. The system loses part of its mass, and the reaction on the bird gives necessary sustentation. The sequence of actions described continues. "Comme partout où va l'oiseau, il est entouré d'air, il trouvera sur sa route un projectile gazeus à utiliser." The writer summarizes his description with the statement, "Et je crois toujours vrai que dans le vol: *tout se passe comme si l'oiseau perdait à chaque battement une partie de sa masse*, ce que je considérais comme le clef du problème." But one can well ask how much this contributes after all to an understanding of the problem. Such a statement as, ". . . le vol animal depende d'impulsions musculaires, fréquentes, courtes et relativement très fortes," leaves one wondering how far the author thought the solution of the problem could be carried.

Different types of flight are considered in the first book, and in connection with each there is a critical analysis of the views of other writers. The work therefore gives far more than the author's own judgment. There is only a page or two of formal mathematics in the book, but there is a little use here and there of some of the basic formulas of air resistance. The later chapters give considerable numerical data concerning aerodynamic properties of the wings of different birds.

The last two books, which are small monographs, give the results, mainly by curves, of experiments made to determine the velocities and power developed by birds and insects. The apparatus employed, as shown by the photographs, is rather elaborate.

K. P. Williams

Leçons sur les Progrès Récents de la Théorie des Séries de Dirichlet. By Vladimir Bernstein. Avec une préface de Jacques Hadamard. (Collection de Monogrammes sur la Théorie des Fonctions). Paris, Gauthier-Villars, 1933. xiv +320 pp.

During the last two decades, many interesting papers have appeared dealing with the relationship between a Dirichlet series and the singularities of the analytic function of a complex variable it represents. The book under review gives for the first time a systematic account of all the results obtained by various authors, allowing the reader to get acquainted quickly with a field where many important questions are as yet unanswered. Following the tradition of the "Collection Borel" in which the book appeared, it is not written exclusively for specialists but also for those mathematicians who wish to approach the subject matter for the first time. All necessary auxiliary methods and theorems are completely discussed, some of them in three appendices avoiding the interruption of the exposition of the theory proper. Very clearly written and complete, this book will be of interest to all mathematicians.

In the first chapter a brief account is given of the classical elementary properties of Dirichlet series. In Chapter 2 the author analyses the structure of the sequence of the exponents (maximal density, index of condensation) and, in connection with the overconvergence of Dirichlet series, its influence on the singularities of the function. Chapter 3 deals with the integral representation of the function $\sum a_n\phi(\lambda_n) \exp(-\lambda_n z)$ in terms of $\phi(z)$ and $\sum a_n \exp(-\lambda_n z)$, that is, with the theorem of Cramer and all its generalizations. Chapter 4, beginning with Landau's theorem on Dirichlet series with positive coefficients, deals with the influence of the frequency of variations of signs among the coefficients. Chapters 5 and 6 are devoted mainly to the work of the author on the relative position of the line of convergence and the line of holomorphy, as well as the frequency of singularities on the latter for series whose exponents have a finite maximal density. Some of the results of these chapters are new and new proofs are given for some known theorems. The reviewer has, however, been unable to follow the proof for the necessity of the existence of the function $\phi(z)$ mentioned in the fundamental theorem on pages 103 and 104. The results obtained in the course of the proof are sufficient, however, to guarantee the truth of all the different applications of this theorem that are made in the sequel. In Chapter 7 the restriction of finite maximal density is dropped and the more or less isolated results which are known for the most general case are taken up. Chapter 8 deals with the composition of singularities, that is, generalizes the classical theorem of Hadamard for power series. With the last chapter we leave the theory proper and consider applications to the general theory of analytical functions, in particular to functions of the exponential type. Some of the theorems obtained are the more interesting since no direct proof for them is known.

The three "notes" at the end of the book are devoted respectively to a complete study of sequences of finite maximal density, to the general properties of the function $\Pi(1-z^2/\lambda_n^2)$, which plays such a fundamental role in the whole theory, and finally to a short account of the transformation of Laplace.

F. Bohnenblust

NOTES

A new quarterly devoted to the publication of mathematical research and sponsored by Duke University will appear shortly under the title *Duke Mathematical Journal*. The editors are A. B. Coble, D. V. Widder, and J. M. Thomas, the last named being managing editor. The associate editors are H. E. Bray, L. W. Cohen, L. R. Ford, J. J. Gergen, R. E. Langer, C. C. MacDuffee, J. A. Shohat, and G. T. Whyburn. The first number will be dated March, 1935. Manuscripts and editorial correspondence may be addressed to the Journal at 4785 Duke Station, Durham, North Carolina; and subscriptions (at $4.00 per year) may be sent to the Duke University Press. Individual members of the Mathematical Association of America may subscribe at half price.

In March, 1935, the Division of Applied Mechanics of the American Society of Mechanical Engineers will begin publication of a quarterly Journal of Applied Mechanics. The editorial board includes J. P. Den Hartog, R. Eksergian, J. Goff, J. C. Hunsaker, G. B. Karelitz, T. von Kármán, A. I. Lipetz, A. Nadai, J. Ormondroyd, R. E. Peterson, S. Timoshenko. The technical editor will be J. M. Lessells, of Swarthmore, Pa.

At an informal meeting held at New York University in connection with a meeting of the American Philosophical Association on December 27, 1934, it was decided to undertake the formation of an association of which one purpose would be the publication of an International Journal of Logic; and a Committee on Organization was appointed. Those who would eventually wish to lend support by joining the proposed association (with dues of probably $3.00 per year) are asked to write to Professor C. J. Ducasse, Brown University, Providence, R.I.

A collection of tables of higher mathematical functions is being prepared at the Institute for Applied Mathematics of the University of Berlin, under the auspices of the Notgemeinschaft der Deutschen Wissenschaft; the work will be done by unemployed mathematicians. Tables of Hermite and Laguerre functions have already been begun.

The Commission on Examinations in Mathematics, whose members are Arnold Dresden (Chairman), Grace S. Barker, Ralph Beatley, William Betz, W. R. Longley, G. R. Mirick, Rolland R. Smith, Anna J. Pell-Wheeler, and Norbert Wiener, was appointed by the College Entrance Examination Board in April, 1933; and has now submitted its Report. Copies of the Report, at ten cents each, may be had by writing Professor T. S. Fiske, Secretary of the Board, 431 West 117th St., New York City. It is desired that the Report be given wide publicity, in order to expedite action and to permit the holding soon of examinations proposed in the Report.

At the annual meeting of the British Mathematical Association, held January 7–8 under the presidency of A. W. Siddons, the following were nominated for election as honorary members: Professors E. Borel, of the Uni-

versity of Paris, G. H. Hardy, of the University of Cambridge, D. E. Smith, of Columbia University, and E. T. Whittaker, of the University of Edinburgh.

The Ackermann-Teubner prize for 1934 has been awarded to Professor E. Trefftz, for his work in applied mechanics.

The prize of the Annali della Reale Scuola Normale Superiore di Pisa (see this Bulletin, vol. 39 (1933), p. 497) has been divided equally between the following mathematicians: Guido Ascoli, of the University of Milan, Pietro Burgatti, of the University of Bologna, and Georges Giraud, of the University of Clermont-Ferrand. All three memoirs were on the theory of partial differential equations of elliptic and parabolic type.

The Belgian Academy has awarded its De Boelpaepe prize to A. Biot, for his work in geometric optics.

The Paris Academy of Sciences announces the award of the following prizes for 1934: the Poncelet prize to Maurice Fréchet, for his mathematical work as a whole; the Francœur prize to Jean Favard, for his work in mathematical analysis; the Montyon prize in mechanics to René Swyngedauw, for his study of transmission belts and ball bearings; the Fourneyron prize to Robert Mazet, for his work in friction; the Henri de Parville prize in mechanics to Jean Leray, for his work in the mechanics of fluids; part of the Binoux prize in geography to Pierre Tardi, for his *Traité de Géodésie*; the LaCaze prize in physics to Eugène Bloch, for his work in physics; the Hébert prize to François Bedeau, for his *Traité de Télégraphie sans Fil* and his research on methods for measurements in high frequencies; the Hughes prize to René Lucas, for his work in electro-optics; the Clément-Félix prize to Marcel Laporte, for his research on the properties of ions and the electric luminescence of gases; the Montyon prize in statistics jointly to Pierre Jéramec, for his statistical work as a whole, and to Louis Potin, for his statistical work as a whole and in particular for his numerical tables; an Henri de Parville prize for a scientific publication to Pierre Sergescu, for his work entitled *Les Sciences Mathématiques*; the Grand Prix of the mathematical sciences to Emil Cotton, for his scientific work as a whole; the Houllevigue prize to Léon Brillouin, for his research and works on modern mathematical statistics; the Jules Mahyer prize to Pierre Humbert, for his work in mathematical analysis; a prize from the Becquerel foundation to Yves Rocard, for his work in the kinetic theory of gases and in optics; a prize from the Frémont foundation to Pierre Vernotte, for his work on the propagation of heat, by conduction or convection.

The Ramanujan Memorial prize, offered by the University of Madras for a thesis submitted by an Indian (or a person domiciled in India) on some definite branch of mathematics, pure or applied, has been divided equally among the following: S. Chandrasekhar, fellow of Trinity College, Cambridge; S. Chowla, reader in mathematics, Andhra University, Waltair, India; D. D. Kosambi, professor of mathematics, Ferguson College, Poona, India.

The annual prize of the American Association for the Advancement of Science has been awarded to Dr. V. A. Knudsen, of the University of Cali-

fornia at Los Angeles, for his paper entitled *The absorption of sound in gases*, read at a joint session of the American Physical Society and the Acoustical Society of America, meeting under Section B (physics) of the Association.

Albert Caquot has been elected a member of the Paris Academy of Sciences in the section of mechanics, as successor to the late Paul Vieille.

The University of Brussels has conferred honorary doctorates on Professors P. Debye, of the University of Leipzig, and R. von Mises, of the University of Istanbul.

Professor A. E. Joliffe, of the department of mathematics at King's College, University of London, has been appointed fellow of the College. He is also an honorary fellow of Corpus Christi College, Oxford, and of Jesus College, Oxford.

The University of London has conferred an honorary doctorate on Dr. Karl Pearson, professor emeritus at that University.

Dr. A. H. Compton, professor of physics at the University of Chicago, was awarded the honorary degree of master of arts on the occasion of the opening of his lectures as George Eastman visiting professor at the University of Oxford.

Dr. Irving Langmuir, of the General Electric Company, has been awarded the Fourth Order of the Rising Sun by the government of Japan.

Professor Frank Löbell, of the Stuttgart Technical School, has been appointed professor of geometry at the Munich Technical School.

Professor Wilhelm Müller, of the German Technical School of Prag, has been appointed professor of mechanics at the Technical School at Aachen.

Dr. Fritz Noether, formerly professor of mechanics at the Breslau Technical School, and Dr. Stefan Bergmann, formerly of Berlin, have been appointed to the faculty of the research Institute of Mathematics and Mechanics at the Kujbyscheff University at Tomsk.

Dr. Hilda Pollaczek-Geiringer, formerly of the University of Brussels, has been appointed associate professor of mathematics at the University of Istanbul.

Professor F. A. Willers, of the School of Mines of Freiburg in Saxony, has retired.

The following transfers have taken place in the Italian Universities: Professor G. Ascoli, from Pisa to Milan (infinitesimal analysis); Professor Enea Bortolotti, from Cagliari to Florence (algebraic analysis); Professor U. Crudeli, from Cagliari to Palermo (mathematical physics); Professor G. Giorgi, from Palermo to Rome (telegraphy and telephony); Professor E. Gugino, from Messina to Palermo (rational mechanics).

Dr. G. Vivanti, professor of calculus in the University of Milan, has retired, having attained the limiting age.

A greatly enlarged program of teaching and research in mathematical statistics is being undertaken at Columbia University this year. Research under the auspices of the Carnegie Corporation is being conducted with Professor Harold Hotelling as director, with a view to clarifying the foundations of statistical methods and extending their scope, and particularly in the development of tests of significance and criteria of accurate estimation. For this work Dr. J. L. Doob has been appointed research associate, and Margaret H. Richards and W. C. Madow research assistants. Professor Felix Bernstein, founder and formerly head of the Göttingen Institute of Mathematical Statistics, is at Columbia University this year as visiting professor of mathematics. A course of training in mathematical statistics has been arranged by the coordination of courses in the departments of mathematics, economics, and astronomy; it includes *Probability*, by B. O. Koopman, *Statistical inference* and *Mathematical economics*, by Harold Hotelling, *Mathematics of heredity and evolution*, by Felix Bernstein, and training in the use of card tabulating and calculating machines, interpolation, and finite differences, by W. J. Eckert.

During the Summer Quarter of 1935 at the University of Chicago, Professors G. A. Bliss and L. M. Graves will conduct a seminar in the calculus of variations, in which Professor E. J. McShane, Dr. M. R. Hestenes, Dr. W. T. Reid, Dr. Max Coral, and others who may be interested, will participate. It is planned to select topics of discussion in accord with the special interests of those who attend.

Professor B. H. Camp has been given funds from the endowment for research at Wesleyan University, to carry on work on problems which have been submitted by the research committee of the American Statistical Association. These are chiefly economic problems for which a statistical analysis is desired.

The announcement in the September, 1934, number of this Bulletin that Assistant Professor Morgan Ward, of the California Institute of Technology, had been appointed to an assistant professorship at the Institute for Advanced Study is erroneous. Professor Ward, and also Professors Alonzo Church, Jesse Douglas, Carl Eckart, C. N. Moore, J. L. Walsh, and Oscar Zariski, are on leave of absence from their respective institutions for the academic year 1934–35 and are in residence at the Institute for Advanced Study, with the title of "worker" at the Institute.

Professor T. H. Hildebrandt has been appointed chairman of the department of mathematics at the University of Michigan, as successor to Professor J. W. Glover. Professor Glover remains in the faculty, as "James Olney distinguished professor."

At Brown University, Max Astrachan has been appointed to an instructorship.

Professor O. D. Chwolson, formerly professor of mathematical physics at the University of Leningrad, died May 11, 1934, at the age of eighty-two.

Dr. Walter von Dyck, formerly professor of mathematics at the Munich Technical School, is dead.

Professor J. G. Gray, of the University of Glasgow, known for his work in the theory and invention of gyroscopes, died November 6, 1934, at the age of fifty-eight.

Sir Horace Lamb, F. R. S., formerly professor of mathematics at the University of Manchester, known for his work in mathematical physics, especially hydrodynamics, died December 4, 1934, at the age of eighty-five.

Professor M. D. Earle, of Furman University, died September 13, 1934.

Professor G. L. Hosmer, of the department of civil engineering at the Massachusetts Institute of Technology, died January 10, 1935, at the age of sixty.

Professor Louis Ingold, of the University of Missouri, died of heart disease at Columbia, Missouri, on January 25, 1935, at the age of sixty-three years. He had been a member of this Society since 1904. He taught in the Department of Mathematics at the University of Missouri in various ranks, beginning in 1905, and was made professor of mathematics in 1924.

Professor J. A. McLaughlin, of St. Bonaventure's College, died in November, 1934.

Sister Marie Cecilia Mangold, professor of mathematics at Trinity College, Washington, D. C., died February 9, 1934.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

103. Dr. E. W. Titt: *$(n-1)$-dimensional characteristic strips and Cauchy's problem for partial differential equations.*

In the present paper the author obtains a Cauchy-Kowalewsky system of first-order partial differential equations equivalent to a second-order partial differential equation in n independent variables. In the case of an analytic equation with analytic data this system of first-order equations solves Cauchy's problem for the second-order equation without the usual transformation of coordinates. The key to the Cauchy-Kowalewsky system is the conception of an $(n-1)$-dimensional characteristic strip. Thus in the case of two independent variables these characteristic strips are one-dimensional. They are in fact the strips of second order found by Goursat, *Leçons sur l'Intégration des Equations aux Dérivées Partielles du Second Ordre*, Hermann, 1896. The idea of an $(n-1)$-dimensional characteristic strip is common to both the first and the second order non-linear partial differential equation. In both cases these strips find application in the problem of determining manifolds for which Cauchy's problem is indeterminate. (Received January 26, 1935.)

104. Mr. Garrett Birkhoff: *Non-commutative integration.*

A constructive theory of integration for functions of a line to a (generally non-commutative) "complete normed vector ring" V is developed, entirely analogous to the author's theory of integration of functions of a space with a Borel ring of measurable sets to a Banach space. The integral is immensely more powerful than the usual product integrals, and specializes to the Denjoy integral (without use of derivatives) in the case V is the real number system. (Received January 19, 1935.)

105. Dr. R. H. Cameron (National Research Fellow): *Linear differential equations with almost periodic coefficients.*

This paper gives new necessary and sufficient conditions that all of the solutions of a system of linear differential equations with almost periodic coefficients should be almost periodic. It also gives new sufficient conditions that a particular solution should be almost periodic. (Received January 24, 1935.)

106. Mr. F. J. Murray (National Research Fellow): *On linear transformations in $\mathfrak{L}_p$, $p>1$.*

In this paper we study closed linear transformations (not necessarily limited) in $\mathfrak{L}_p$, by the device exploited by J. von Neumann, for $\mathfrak{L}_2$, of considering them as defined by a closed linear manifold in the product space $\mathfrak{L}_p \times \mathfrak{L}_p$, with certain obvious properties. The relationships between the operations of forming an adjoint, closure in a space with lower index, and intersection with a space of higher index, are obtained. It is possible to define a symmetric relationship like the one which holds for Hermitian operators in $\mathfrak{L}_2$. This is discussed in terms of the above results. Finally, we consider the closure in $\mathfrak{L}_p$, $1<p<2$, of transformations in $\mathfrak{L}_2$. General results are obtained in terms of the adjoint. We then consider projections and more specific results are obtained in terms of the closure of the ranges. Examples are given to illustrate how these results may be applied to the study of a particular transformation and to show that certain results are the best obtainable. (Received January 26, 1935.)

107. Dr. W. T. Martin (National Research Fellow): *Linear difference equations with arbitrary real spans.*

In the first part of this paper we consider difference equations of the form (1) $\sum_{s=0}^{m} c_s(z)\ F(z+\delta_s)=G(z)$, in which the δ_s are any positive numbers and the $c_s(z)$ are analytic functions, asymptotically constant in a given sector. The methods developed by Bochner (Mathematische Zeitschrift, vol. 33 (1931), pp. 426–450) in his treatment of equations of the form (1), where the functions are functions of a real variable x, are applicable to the case where the functions are analytic. We first study the equation (1) with constant coefficients and, using the results thus obtained, we next treat by successive approximations the equation (1) with asymptotically constant coefficients. In the second part of the paper we study systems of the form (2) $\sum_{k=1}^{p}\sum_{s=0}^{m_{jk}} c_s^{jk}(z)\ F_k(z+\delta_s^{jk})=G_j(z)$, $j=1, \cdots, p$. A symbolic method of the sort developed by Bochner (Mathematische Annalen, vol. 104 (1931), pp. 579–587) and Carmichael (Transactions of this Society, vol. 35, (1933), pp. 1–28) is used in the treatment of the system (2) with constant coefficients. Again, a method of successive approximations is used to treat the system (2) with asymptotically constant coefficients. (Received January 26, 1935.)

108. Professor Cornelius Lanczos: *A new approximation method in solving linear differential equations with non-oscillating coefficients.*

The method presented is based on the following principles. (1) The Hamiltonian canonical form of a linear differential equation in connection with the canonical transformations permits us to separate the periodic or exponential part of the solution, reducing the problem to the determination of a mere amplitude factor (see Annalen der Physik, vol. 20 (1934), pp. 635–688). (2) This amplitude factor is a slowly changing function which can be efficiently approximated by a small number of powers of x, either ascending or descending, according to the given range. (3) The powers of x in the real domain

$x=0$ to $x=1$ are highly non-orthogonal, that is, x^{n+1} can be practically replaced by a linear combination of the lower powers. The Taylor series and the ordinary semi-convergent power series expansions approximate efficiently only in the neighborhood of a point. The method here developed distributes the error evenly over a large range and allows us to approximate by a small number of powers much more efficiently, in accordance with the ordinary approximations by means of orthogonal function systems. The coefficients of the expansion, however, are here determined by simple recurrence formulas. (Received January 26, 1935.)

109. Dr. S. E. Warschawski: *On the angular derivatives of univalent functions.*

Let R be a simply connected region in the w-plane, $w=u+iv$, whose boundary contains $w=0$. For $|v|\leqq a$, $a>0$, let $u=f(v)$, with $f(0)=f'(0)=0$, represent an analytic arc, and $\delta(v)\geqq 0$ be a continuous function such that (i) $\delta''(v)$ is bounded, (ii) $\int_{-a}^{a}\delta(v)\,dv/v^{n+1}$, $n\geqq 1$, exists, and (iii) the boundary of R in a neighborhood of $w=0$ is between the curves $u=f(v)-\delta(v)$ and $u=f(v)+\delta(v)$, the latter being inside R. We prove that if $z=z(w)$ maps R conformally on $|z|<1$, $\lim_{w\to 0}z^{(\nu)}(w)=z^{(\nu)}(0)$ exists for $1\leqq\nu\leqq n$ in any angle (*) $|\arg w|\leqq\alpha<\pi/2$ within R, and also (*) $z(w)=z'(0)\,w+(1/2)z''(0)\,w^2+\cdots+(1/n!)z^{(n)}(0)\,w^n+r_n(w)w^n$, where $\lim_{w\to 0}r_n(w)=0$ in (*). If (i) is replaced by the hypothesis that $\delta(v)$ increases monotonically with $|v|$, the expansion (*) holds in a wider region than (*). For $n=1$ the latter result extends a theorem of C. Carathéodory and its generalizations by C. Visser, J. G. van der Corput, and the author. (Received January 26, 1935.)

110. Mr. D. L. Netzorg: *On positive Cotes' numbers.*

For given $x_1,\cdots,x_n$, $(x_1<\cdots<x_n)$, there is a mechanical quadrature formula $\int_{-1}^{1}f(x)\,dx=c_1f(x_1)+\cdots+c_nf(x_n)$, valid for all polynomials of degree less than n. We assume that the c's (which are the Cotes' numbers corresponding to the x's) are all positive. This entails that one, at least, of the x's lie between any two successive roots of $Q(x)=P_m(x)+aP_{m-1}(x)$, where $2m\leqq n+1$, $P_m(x)$ is the mth Legendre polynomial, and a is real. Furthermore x_1 is less than q_1, the smallest root of $P_{m-1}(x)$, while x_n is greater than its largest root. As to the Cotes' numbers themselves we find that if $r_i<x_j<r_{i+1}<r_{i+2}$, then $1+r_i<H_1+\cdots+H_i<c_1+\cdots+c_j<H_1+\cdots+H_{i+1}<1+r_{i+2}$, where the r's are roots of $Q(x)$, and the H's are the Cotes' numbers corresponding to the r's. Furthermore, if $x_j<q_1$, then $c_1+\cdots+c_j<G_1$, where the G's are Cotes' numbers corresponding to the roots of $P_{m-1}(x)$. From these facts it is easy to prove a result due to Fejér, obtained by him in a different fashion (Mathematische Zeitschrift, vol. 37 (1933), pp. 287–309). For any positive weight function similar results hold. (Received January 28, 1935.)

111. Dr. A. C. Offord: *On Fourier transforms*, III.

This paper is a continuation of an earlier investigation (London Mathematical Society Proceedings, (2), vol. 38 (1934), pp. 197–216). A condition for a function $f(x)$ to have a Fourier transform $F(x)$ in the class $L^p(1\leqq p\leqq\infty)$ is

given and an expression for $f(x)$ in terms of $F(x)$ is found. Conversely, a condition is obtained for a function $F(x)$ of $L^p(p>2)$ to be the Fourier transform in L^p of some function $f(x)$. (Received January 30, 1935.)

112. Professor Otto Szász: *Converse theorems of summability for Dirichlet series.*

This paper contains simplifications and generalizations of some "Tauberian" theorems for Dirichlet series. Let $F(t)=\sum_1^\infty c_\nu e^{-\lambda_\nu t}$ converge for $t>0$ and let the limit $\lim_{t\to 0}F(t)=s$ exist. We derive various sufficient conditions for the convergence (to the value s) of the series $\sum_1^\infty c_\nu$. These conditions are more general than those known before. An essential feature of our conditions is that the customary hypothesis $\lambda_{\nu+1}/\lambda_\nu\to 1$ is entirely eliminated. (Received February 19, 1935.)

113. Professor C. R. Adams and Dr. Hans Lewy: *On convergence in length.*

Let $f_n(x)$ $(n=1, 2, 3, \cdots)$ be a sequence of functions of bounded variation converging on $a\leqq x\leqq b$ to a limit function $f_0(x)$ of bounded variation; let the length in the sense of Peano [total variation] of f_n on $a\leqq x\leqq b$ be denoted by $L(f_n)$ [$T(f_n)$] $(n=0, 1, 2, \cdots)$; then f_n is said to converge in length [variation] to f_0 on (a, b) when and only when we have $L(f_n)\to L(f_0)$ [$T(f_n)\to T(f_0)$]. Concerning convergence in variation see Adams and Clarkson, this Bulletin, vol. 40 (1934), pp. 413–417. The present paper (i) shows that convergence in length implies convergence in variation (the converse is not true even when f_n, $(n=0, 1, 2, \cdots)$, is assumed absolutely continuous); (ii) examines the relationship between convergence in length and uniform convergence; (iii) shows that convergence in length, when f_0 is absolutely continuous, implies $T(f_n-f_0)\to 0$; (iv) proves that convergence in length is invariant under addition and multiplication of sequences when the limit functions are absolutely continuous; and (v) shows that convergence in length need not be invariant under addition of two sequences when both limit functions are merely continuous even though all approximating functions are absolutely continuous. (Received February 14, 1935.)

114. Professor J. L. Walsh: *Orthogonal polynomials and functions of the second kind.*

Let $n(z)$ be positive and continuous on the rectifiable Jordan curve C, let $\{p_k(z)\}$ be the polynomials normal and orthogonal on C with respect to $n(z)$, and let $\{q_k(t)\}$ denote the corresponding functions of the second kind: $q_k(t)\equiv\int_C [n(z)p_k(z)\,|dz\,|]/(t-z)$, t exterior to C. Let $f(z)$ be analytic in the closed annulus A bounded by C and C_R (Kreisbild for the mapping of the exterior of C). We have, interior to A, uniformly on any closed set interior to A, $f(z)=(1/(2\pi i))\sum_{k=0}^\infty p_k(z)\int_\Gamma f(t)\,q_k(t)\,dt-(1/(2\pi i))\sum_{k=0}^\infty q_k(z)\int_\Gamma f(t)\,p_k(t)\,dt$, where Γ is an arbitrary contour in A containing C in its interior. (Received February 11, 1935.)

115. Professor M. Fekete: *Some generalizations of Paley's theorems on Fourier series with positive coefficients.*

This paper contains various generalizations of recent theorems of Paley, where the conditions of positiveness of the coefficients is replaced by conditions of the type $a_n+\alpha_n \geqq 0$, $b_n+\beta_n \geqq 0$ where α_n, β_n are terms of series which are "slowly oscillating" in a certain sense. Necessary and sufficient conditions are obtained for uniform convergence of the Fourier series of a continuous function, or for uniform boundedness of partial sums of the Fourier series of a bounded function. (Received February 23, 1935.)

116. Professor Malcolm Foster: *Congruences with a common middle envelope.*

Relative to the moving trihedral on the unit sphere S, let (a, b) and $(\bar{a}, \bar{b})$ be two points in the tangent plane through which two lines l and $\bar{l}$ are drawn parallel to the normal to S. The necessary and sufficient condition that the congruences formed by the lines l and $\bar{l}$ have a common middle envelope is that the congruence $(a-\bar{a}, b-\bar{b})$ has the center of S for its middle envelope. The paper considers the case of a common middle envelope for two normal congruences, and for the congruences defined by the points (a, b) and $(-b, a)$. (Received January 25, 1935.)

117. Miss Jeanette Fox: *Finiteness of the number of quadratic fields with even discriminant and Euclid algorithm.* Preliminary report.

The only quadratic fields $R(m^{1/2})$, m even, which have a Euclid algorithm are $R(2^{1/2})$ and $R(6^{1/2})$. For quadratic fields $R(m^{1/2})$, $m \equiv 3 \pmod 4$, there are at most four fields with Euclid algorithm. Additional results have been obtained for fields with odd discriminant. (Received January 26, 1935.)

118. Dr. D. H. Lehmer: *On Lucas's test for the primality of Mersenne's numbers.*

In 1930 the writer proved as a by-product of a long discussion that the sufficient condition of Lucas for the primality of $2^{4k+1}-1$ is also necessary and equally applicable to $2^{4k-1}-1$. In fact, we may state that a necessary and sufficient condition for the primality of $N=2^n-1$ (n is odd) is that N divide the $(n-1)$st term of the sequence 4, 14, 194, $\cdots$ in which each term is the square of its predecessor minus 2. In 1932 A. E. Western proved this theorem by means of the theory of algebraic numbers. In this paper the writer presents a very elementary yet self-contained proof of this test for primality. (Received January 25, 1935.)

119. Professor I. J. Schoenberg: *Note on metric geometry.*

The problem of constructing simplexes with assigned lengths for the edges in (definite or indefinite) euclidean spaces and in spherical spaces, studied recently by Menger, Blumenthal, Wald, Coxeter, Todd, and Klanfer, is very simply solved by means of quadratic forms. Thus for the sphere $S_r^2: x_1^2+x_2^2 + \cdots +x_{r+1}^2=\rho^2$ we have the following theorem. Let α_{ik}, $(i \neq k; i, k=1, \cdots, n; \alpha_{ik}=\alpha_{ki})$, be $n(n-1)/2$ given positive quantities. A necessary and sufficient

condition that there be, on some spherical space of radius ρ, n points A_1, $A_2, \cdots, A_n$ of mutual (shortest) spherical distances $A_iA_k=\alpha_{ik}$, $(i\neq k)$, is that the quadratic form $F=\sum_{i,k=1}^{n}\cos(\alpha_{ik}/\rho)\,x_ix_k$, $(\alpha_{ii}=0)$, be always $\geqq 0$. If r is the rank of F, then we can find such points in S^{ρ}_{r-1} but not in S^{ρ}_{r-2}. (Received January 28, 1935.)

120. Professor Oscar Zariski: *On the Poincaré group of plane algebraic curves.* First communication.

The paper deals with the Poincaré group G of the residual space of the dual C of any plane algebraic curve Γ with ordinary double points and of any other curve of which C is a limiting case. Use is made of the following lemma: *the Poincaré group of an algebraic hypersurface V in S_r with respect to its carrying S_r coincides with the Poincaré group of a generic plane section of V with respect to its carrying plane.* In the present communication the case where Γ is rational is considered. If Γ is of order n, C is the plane section of the discriminant hypersurface in an S_n. It is found that G possesses $n-1$ generators $g_1, g_2, \cdots, g_{n-1}$ (Hurwitz) and the generating relations are: (a) $g_ig_j=g_jg_i$, if $i-j\neq\pm 1$; (b) $g_ig_{i+1}g_i=g_{i+1}g_ig_{i+1}$; (c) $\prod_{i=1}^{n-1}g_i\prod_{i=n-1}^{1}g_i=1$. It is then proved that any plane curve $\overline{C}$ which has C as a limiting case but has less or simpler singularities than C possesses a cyclic Poincaré group. In particular, *the Poincaré group of any rational curve of even order, other than C, is cyclic.* The case where C is a sextic is an exception. (Received February 23, 1935.)

121. Dr. C. C. Grove: *A note on the word mantissa.*

The statements in most dictionaries on this word led me to original sources and uses because there were some manifest misstatements. The word first appeared in mathematics in its true etymological sense in the Latin edition of the *Algebra* of Johannis Wallis, 1693, p. 41: Ejusque partes decimales abscissas, 0.12003416, *Appendicem* voco, sive *Mantissam fractionis primae*, seu *Mantissam primum*. This arose in writing out the successive convergents of the continued fraction for $2684769/837.6571=1/3.12003416+$. Two chapters farther on he does not use the word in writing on logarithms. Leonardus Euler in his *Introductio in Analysin Infinitorum*, 1748, may conceivably have etymologised mantissa popularly as a Greek feminine of μάντις "prophet" in saying "Sola ergo Mantissa indicabit figuras numerum componentes." He thus introduced the use of the word in logarithms. Finally, C. F. Gauss in *Disquisitiones Arithmeticae*, 1801, goes back somewhat towards its original use, in Art. 312, p. 543: "Definitio. Si fractio communis in decimalem convertitus, seriem figurarum decimalium fractionis mantissam vocamus, ... Ita e.g. fractionis 1/8 mantissa est 125, mantissa fractionis 35/16, 1875, fractionis 2/37 mantissa $054054\cdots$ in inf." (Received February 14, 1935.)

122. Professor H. A. Simmons: *The classification and general solution of certain diophantine problems which involve special systems of equations of the second degree.*

The main result of the paper, expressed in Theorems 2 and 6, is essentially as follows: consider the mixed system of diophantine equations and inequali-

ties T: $a_1(n)x_1-a_j(n)x_j+b_j(n)=0$, $(j=2, \cdots, m)$, $x_1 \geqq x_2 \geqq \cdots \geqq x_m>0$, $n>0$, $a_1(n) \geqq a_2(n) \geqq \cdots \geqq a_m(n)$, where the $a_i(n)$, $(i=1, \cdots, m)$, $b_j(n)$ are functions of n that take only positive integral values for positive integral values of n and the inequality sign holds between at least one pair of the a's (for every positive integer n). Suppose T is solvable when n is assigned a particular positive integral value, n_0, and let T_0 be the system gotten by setting $n=n_0$ in T. System T_0 has only a finite number of solutions. If also there is only a finite number of values of n for which T has a solution (as is the case in §3 of the paper), then T, considered as a system with n and the x's as unknowns, has only a finite number of solutions. Let T' be the system that is obtained from T by merely reversing the inequality signs between the a's. Then if n_0 is a value of n for which T' is solvable, T' has infinitely many solutions associated with n_0 (as is the case in certain examples of §4 of the paper). (Received February 6, 1935.)

123. Professor Raymond Garver: *On Bieberbach's theory of cubic constructions.*

In vol. 167 of the Journal für Mathematik, Bieberbach solves the problem of cubic constructions with the aid of a very simple use of a movable right angle. Constructions of this sort are, of course, known. The present paper studies Bieberbach's method of trisection, proves it equivalent to trisection with the aid of a limaçon, gives an alternate form not using a right angle but a different type of motion, and applies the method to certain particular constructions. (Received January 28, 1935.)

124. Professor C. N. Moore: *On convergence factors in multiple series summable by Nörlund means.*

The purpose of the present paper is to generalize to multiple series the theorem concerning convergence factors in simple series given in a paper presented at the annual meeting in Pittsburgh (see this Bulletin, abstract 41–1–56). The conditions obtained include the analogues of those obtained in the previous paper and certain supplementary conditions, such as usually arise in extending convergence factor theorems to multiple series. (Received January 26, 1935.)

125. Professor Edward Kasner: *A new inversive invariant.*

If we consider two curves touching each other (horn angle), a simple projective invariant is the ratio of the two curvatures γ_1 and γ_2 at the point of contact (Mehmke's theorem). The present author obtains a differential invariant for the inversion group, namely $I_3=(d\gamma_1/ds_1-d\gamma_2/ds_2)/(\gamma_1-\gamma_2)^2$. This is a combination of the curvatures and the deviations (or rates of change of curvature with respect to arc length). This fundamental quantity I_3 is not only an *inversive* invariant, it is also a *conformal* invariant. (See Kasner, *Conformal geometry*, Cambridge International Mathematical Congress Proceedings, 1912, vol. 2, p. 84.) (Received January 15, 1935.)

126. Professor J. A. Shohat: *On mechanical quadratures.*

The author tries to approximate $F(h)=\int_a^{a+h} f(x)dx$ by an expression of the type $\Psi(h)=h\sum_{i=0}^{k-1} b_i f(a+n_i h)$, $0\leqq n_i\leqq 1$, where the constants b_i (independent of h) and n_i are so chosen that $F(h)-\Psi(h)=O(h^{2k+2})$. There is thus obtained a mechanical quadratures formula either of ordinary Gauss type or with one or both end abscissas fixed at the end points of the interval of integration. *The interior points turn out to coincide with the zeros of a certain class of Jacobi polynomials.* It is shown that the classical trapezoidal and Simpson formulas of mechanical quadratures are a special case, and thus their remainder is immediately derived. Some general properties of mechanical quadratures formulas with fixed abscissas are then discussed, and the asymptotic behavior of some of their coefficients for $n\to\infty$ is described. (Received January 17, 1935.)

127. Professor A. D. Campbell: *Pseudo-covariants in a Galois field and their algebraic and geometric applications.*

In this Bulletin for April, 1933, the author discussed briefly certain collections of terms from n-ics in m variables in a Galois field that have the peculiarity of being irreplaceable by any linear transformation if they are missing from the original n-ic. These sets of terms were called pseudo-covariants The present paper contains a more precise and thorough discussion of these pseudo-covariants as to their number for any n-ic, their exact definition, their order of disappearance. Also a study is made of the algebraic and geometrical meanings for the n-ic of the identical vanishing of these pseudo-covariants. (Received January 24, 1935.)

128. Mr. Max Astrachan: *Studies in the summability of Fourier series by Nörlund means.*

The author gives sets of sufficient conditions on the generating sequence $\{p_\nu\}$ in order that (I) the Fourier series and the conjugate Fourier series of $f(x)\subset L$ be summable by the Nörlund method (N, p_ν) to the correct value at all points where $\phi_1(t)=\int_0^t[f(x+s)+f(x-s)-2f(x)]\,ds=o(t)$, and $\int_0^t[f(x+s)-f(x-s)]\,ds=o(t)$, respectively; (II) the rth derived Fourier series be summable to the rth generalized derivative of $f(x)$ (in the sense of de la Vallée-Poussin) whenever this derivative exists; (III) the rth derived conjugate Fourier series of $f(x)$ be summable. The conditions, when (N, p_ν) is specialized to be (C, α), imply the restrictions that for (I) $\alpha>1$, (II) $\alpha>r$, (III) $\alpha>r+1$. If the method (N, p_ν) be superimposed on $(C, 1)$ the Fourier series is summable to $f(x)$ at all points where $\phi_1(t)=o(t)$ provided the generating sequence $\{p_\nu\}$ satisfies the Fourier-effective conditions of Hille-Tamarkin. (Received January 16, 1935.)

129. Mr. Garrett Birkhoff: *Abstract continuous groups.*

An axiomatic treatment of abstract continuous groups is given, based on properties of linear deformations of Banach space. The definition specializes to that of abstract Lie groups in case the infinitesimal generators have a finite basis, and to that of Banach spaces in case the groups are Abelian. The theory is then used as a tool in a general investigation into the structure of groups of deformations. (Received January 19, 1935.)

130. Dr. L. A. Dye: *Involutorial transformations associated with a rational ruled surface.* Part I.

In this part of the paper a (1, 1) correspondence is established between a pencil of quadric surfaces and the generators of a ruled surface R_{n+1} of order $n+1$ with an n-fold line l. A point P in space determines a quadric of the pencil and a generator r of R_{n+1}. The plane determined by P and r is tangent to R_{n+1} at a point Q. The line PQ meets the quadric in a residual point P' which is the image of P in an involutorial space transformation. Two cases are considered here: (1) the pencil of quadrics have a quartic space curve of the first species in common; (2) the pencil of quadrics have the line l and a space cubic as a basis curve. Some interesting contact conditions are involved in the homaloidal webs of surfaces arising from these transformations. (Received January 23, 1935.)

131. Professor Edward Kasner: *The dual of Lie's contact transformation of dynamics.*

The infinitesimal contact transformations of dynamics, first studied by Lie, have a characteristic function of the form $W=\Omega(x, y)\ (1+p^2)^{1/2}$. They may be characterized by the fact that the path of any element (x, y, p) is an orthogonal series of elements. A generalization introduced by the author arises by requiring the path series to be *isogonal* series. The ∞^2 path curves form a generalization of natural families. A dual type of infinitesimal contact transformation is obtained by demanding that the path series shall be *equitangential*. The characteristic function $W(u, v, w)$, where w is dv/du, is obtained in explicit form: $W=e^{-w/c}\ \Omega(u, v)$. The ∞^2 paths are studied; the ∞^2 envelope curves form a new type of family which may be regarded as the dual of the natural family. A further generalization is obtained by replacing the constant c by any function of the line coordinates (u, v). (Received January 15, 1935.)

132. Professor Edward Kasner: *Osculating turbines.*

The concept of geometric *turbine* was introduced by the author in a paper in the American Journal of Mathematics, 1911. A turbine consists of ∞^1 lineal elements whose points are on a circle and whose directions are equally inclined to the circle. In the present paper general *series* of ∞^1 elements are studied. At each element there is a definite osculating turbine. The set of ∞^1 osculating turbines defines a field, that is, a specific new type of differential equation of first order. The integral curves are termed the *related curves* of the given series. If a set of ∞^1 turbines is given, we obtain a criterion as to when it may be considered as an osculating set. The whole theory is a wide extension of the standard theory of osculating circles of curves. (Received January 16, 1935.)

133. Dr. Solomon Kullback: *A note on the analysis of variance.*

In a paper in this Bulletin, vol. 38 (1932), pp. 731–735, H. L. Rietz considers a set of independent items classified in some relevant manner into N sets of S items each. By making use of a dispersion theorem of Coolidge (this Bulletin, vol. 27 (1921), p. 439), Rietz arrives at estimates of variance used by

R. A. Fisher without making use of arguments involving the number of degrees of freedom of the items concerned. In this note we employ the same procedure for a set of independent items classified into N sets of $S_i (i=1, 2, \cdots, N)$ items each and arrive at estimates of variance used in discussions concerning the correlation ratio and multiple correlation coefficient. (Received January 19, 1935.)

134. Professor Marie Litzinger: *A basis for residual polynomials in n variables.*

Kempner established the existence of a basis for residual polynomials in one variable with respect to a composite modulus (Transactions of this Society, vol. 22 (1921), pp. 240–266), and considered the problem for n variables (Transactions of this Society, vol. 27 (1925), pp. 287–298). Dickson in his *Introduction to the Theory of Numbers*, pp. 21–27, developed a brief method of obtaining Kempner's results for polynomials in one variable. The purpose of this paper is the application of Dickson's method to polynomials in n variables. (Received January 22, 1935.)

135. Professor T. H. Rawles: *Operational methods in theoretical population problems.*

In this paper is discussed the state of a population which is subject to the following laws: (a) in each generation one member dies and another divides; (b) new members are of the same kind as the members whose division produced them. It is shown that the mean number of generations required for the entire population to reach a state where all members are the descendants of a single one of the original ones is $(N-1)^2$, where N is the number of members in the population. (Received January 19, 1935.)

136. Dr. W. T. Reid: *A certain three-dimensional continuum.*

In this note there is given an example of a bounded continuum in three-dimensional euclidean space such that there exists a point A of M and subcontinua G, K_1, and K_2 of $M-A$ satisfying the following conditions: (1) $M-K_i$, $(i=1, 2)$, is the sum of two mutually separated point sets each of which is connected, (2) each point of G is separated from A in M by either K_1 or K_2; however, (3) there does not exist a point set consisting of a finite number of connected sub-sets of M and separating A from G in M. (Received January 25, 1935.)

137. Dr. M. S. Robertson (National Research Fellow): *On the coefficients of a typically-real function.*

The class of functions $f(z)=\sum_1^\infty a_n z^n$ which are "typically-real" in the unit circle, that is, $f(z)$ is regular for $|z|<1$, a_n real, and $If(x+iy)>0$ when $y>0$ for $x^2+y^2<1$ (I denoting "the imaginary part of"), are considered. In particular, if $f(z)$ is regular, real on the real axis, and univalent for $|z|<1$, then $f(z)$ is also typically-real. It is shown that $E(r)\equiv\int_{-1}^{1}(f(rt)/t)dt$ is the greatest oscillation of the real part of the function $\int_0^z (f(z)/z)dz$ on $|z|=r$, and that $E=\lim_{r\to 1} E(r)$ exists, finite or infinite. If E is finite, then $|a_n|\leqq 2E/\pi$ for all n

and the equality sign is attained for n odd by the function $z/(1+z^2)$. (Received January 24, 1935.)

138. Professor Otto Szász: *Generalization of two theorems of Hardy and Littlewood on power series.*

Hardy and Littlewood have proved the following theorems: If $\lim_{x\to 1-0} \sum_0^\infty a_\nu x^\nu = s$ and (1) $na_n \geqq -k$, $k>0$, $n=1, 2, 3, \cdots$, then $\sum a_\nu$ converges to s; the same holds with the condition (2) $\sum \nu^\rho |a_\nu|^{\rho+1} < \infty$, $\rho>0$, instead of (1). The author replaces (1) and (2) respectively by $\sum_1^n \nu^\rho (|a_\nu| - a_\nu)^p = O(n)$ and $\sum \nu^\rho (|a_\nu| - a_\nu)^{\rho+1} < \infty$, giving simple proofs for these generalizations. (Received January 26, 1935.)

139. Mr. M. S. Webster: *The zeros of Jacobi polynomials.*

The main results of this paper may be divided into two parts. In the first part the author obtains, by a lengthy but elementary analysis, the number and location as far as the intervals $(-\infty, -1)$, $(-1, 1)$, $(1, \infty)$ are concerned of the real zeros of the Jacobi polynomial $J_n(x; \alpha, \beta)$ where the parameters α, β are arbitrary real numbers (not necessarily positive). The second part, with α, β positive, contains upper and lower bounds for the said zeros which in this case are all inside the interval $(-1, 1)$. A geometric method employed by E. R. Neumann and C. Winston for Laguerre and Hermite polynomials is used. Applications are made to mechanical quadratures. (Received February 4, 1935.)

140. Professor C. G. Latimer: *On ideals in generalized quaternion algebras and Hermitian forms.*

The elements of a generalized quaternion algebra may be written in the form $x+yj$, where x, y are in a quadratic field F, $j^2=-\alpha$ a rational integer and $yj=jy'$, y' being the conjugate of y. Let G be the set of all such elements with x, y algebraic integers. It is shown that there is a one-to-one correspondence between certain classes of ideals in G and the classes of Hermitian forms over F of determinant $-\alpha$. Also there is proved a theorem on the existence of a g.c.d. and on factorization in G under the assumption that there is a single class of such forms. A number of results in the literature on special quaternion algebras may be deduced immediately from these results. There are also obtained for an infinitude of algebras the same theorems on the existence of a g.c.d. and on factorization in G as were obtained by Dickson for the Lipschitz integral quaternions (*Arithmetic of quaternions*, Proceedings of the London Mathematical Society, (2), vol. 20 (1922), pp. 225–232, Theorems 3, 8). (Received January 25, 1935.)

141. Dr. Solomon Kullback: *A note on the multiple correlation coefficient.*

In this note, there is derived the expected values of certain estimates of variance employed in R. A. Fisher's method of the analysis of variance of the multiple correlation coefficient. The results are established without making use of arguments involving the number of degrees of freedom of the items con-

cerned. The method of procedure employs the theory of characteristic functions and makes use of an integral whose value has been established by S. S. Wilks. (Received February 5, 1935.)

142. Mr. H. M. MacNeille: *Extensions of partially ordered sets.* Preliminary report.

Elements $a, b, c, \cdots$, and a relation $\subset$ satisfying postulates (1) $a \subset b$ and $b \subset c$ imply $a \subset c$ and (2) $a \subset a$ form a partially ordered set. In terms of the given elements and relation, new relations, $\supset$ and $=$, are defined, and certain elements, not always present, are characterized as units, complements, sums, products, modular sums, modular products, distributive sums, and distributive products. Partially ordered sets are classified according to the characterized elements they contain. The classification includes lattices, complete lattices, modular lattices, distributive lattices, and Boolean algebras. The classifications, except those shown to be equivalent, are distinguished by examples. The equivalence of these definitions with the usual ones, for such classifications as Boolean algebras and the various types of lattices which are customarily defined in other ways, is established. The object of the extensions treated in this paper is to adjoin elements, as few as possible, to a given set, so that the resulting set belongs to a more specialized classification. The adjunction of units is immediate. Sums and products, of any number of elements, are adjoined by suitably defined cuts. Distributive sums or products are provided by adjoining ideals, and complements, by adjoining finite descending sequences. These extensions are established and applied. (Received February 2, 1935.)

143. Professor G. D. Birkhoff and Dr. M. R. Hestenes: *Oscillation, separation, and comparison theorems in the calculus of variations.*

We consider the accessory minimum problem described in abstract 41-3-173 and define the numbers m, r, s, as in that abstract. We suppose the strengthened condition of Legendre holds. The maximum number of admissible variations satisfying the end conditions and having $J_2(\eta) < 0$ for every proper linear combination of these variations is the same for every such set and is equal to m. There are exactly $m-r$ conjugate points of x_1 between x_1 and x_2. For two different functionals J_2, J_2^* subject to different end conditions we always have $m-m^*=r-r^*+s-s^*$. Each conjugate family $\mathfrak{F}$ belongs to some functional $J_2(\eta)$ and hence has an order of concavity r, namely that of $J_2(\eta)$. If the number of focal points of $\mathfrak{F}$ between x_1 and x_2 is equal to m, then $m=r+s$. For two conjugate families $\mathfrak{F}$, $\mathfrak{F}^*$ we have $m-m^*=r-r^*$. The numbers r and r^* are easily computed and are each less than n, where $n+1$ is the dimensionality of our problem. From these results one can obtain as immediate corollaries previous oscillation, separation, and comparison theorems of this type. Applications have also been made to periodic extremals. We do not distinguish between the degenerate and non-degenerate cases. (Received February 18, 1935.)

144. Philip O. Bell: *Tetrahedra associated with canonical developments for the equation of a curved surface.*

A number of different canonical developments for the equation of a curved surface have been obtained by investigators in the field of projective differential geometry. These developments are simplest in form when the vertices of the associated tetrahedra are located on a certain quadric, known as the canonical quadric of Wilczynski. The geometrical location of this quadric was first accomplished by Wilczynski. Bompiani has offered a distinctly different definition, and more recently Stouffer has obtained a simple method of locating the quadric. The canonical quadric is actually useful in locating only one of the four vertices of the tetrahedron. It is the purpose of this paper to locate this fourth vertex for the whole series of expansions by rather elementary methods and without the introduction of Wilczynski's quadric. Incidentally, the quadric is located as soon as any one of these fourth vertices is determined. (Received March 2, 1935.)

145. Professor H. Bateman: *The problem of the shopper.*

Let $C(m, n)$ be the chance that at any shop a shopper spends n out of m cents in his possession. If he has p cents originally the chance $W(r, s)$ that he has r cents left after visiting s shops may be found by means of a recurrence relation. If, however, $G(r, t)$ is the generating function of the functions $W(r, s)$, a set of linear equations may be used for the determination of the functions $G(r, t)$. The recurrence relation may be solved also by means of a contour integral and in this paper a study is made of the functions which arise when special assumptions are made concerning the quantities $C(m, n)$. (Received March 5, 1935.)

146. Mr. D. L. Webb: *Generation of an n-valued logic by one binary operation.*

The author shows that any function of two elementary propositions in an n-valued logic can be generated by the iteration of one binary operation. This is accomplished by finding functions of two propositions in terms of this binary operation with propositions such that they can be combined by another binary operation, defined in terms of the first, so as to give any function of two elementary propositions defined by the truth table of an n-valued logic. (Received March 7, 1935.)

147. Mr. V. Elconin: *Primitive functions on a finite set.*

Let E be a set of n distinct elements, and let $p(u, v)$ be in E for any u, v in E. The "iterations" $f, g, h, \cdots$ of p are the functions of u or v or u, v with the property that any f is identically $p(g, h)$, or $p(g, b)$, or $p(a, h)$ or $p(a, b)$, for some g, h, a, b, where a, b are either u or v. Here $p(u, v)$ is "primitive" if and only if any function of u, v with values in E is an iteration of $p(u, v)$. It is shown in this note that the number of primitive functions is at least $n!$. From this it follows that the truth-functions of two variables in an n-valued logic can be generated from a single function, which is a generalization of Sheffer's well known result for $n=2$. (Received March 7, 1935.)

148. Professor A. D. Michal: *A reduced set of postulates for "Riemannian" differential geometry in abstract vector spaces.*

In two former papers (abstracts 39-11-332, 333) the author gave a set of postulates and developed a theory of parallel displacement and curvature for Riemannian differential geometry in Banach spaces. Since then several new theorems on differentials and total differential equations in abstract vector spaces have been proved. These theorems, among others, are used in the present paper to obtain a drastically reduced set of postulates. (Received March 7, 1935.)

149. Professor A. F. Moursund: *A note on Taylor's theorem.*

In this note, which is soon to appear in this Bulletin, we generalize the descriptive form of Taylor's theorem given by S. Pollard (Cambridge Philosophical Society Proceedings, Vol. 23 (1926–27), pp. 383–385). We consider only functions of one variable. (Received February 9, 1934.)

150. Professor Gordon Pall: *A new solution of the Gauss problem on* $h(s^2d)/h(d)$.

Let $h(d)$ denote the number of classes of primitive integral b.q.f.'s of discriminant $d \neq 0$. The formula $h(p^2d)/h(d) = \{p-(d\,|\,p)\}/\sigma$ is here proved by a new method, based on comparing the number of sets of representations of n and p^2n in the forms of discriminants d and p^2d. (Received February 11, 1935.)

151. Dr. J. M. Feld: *Configurations inscriptible in a plane cubic curve.*

This paper is concerned with the inscriptibility of certain configurations in real non-singular plane cubics, and with properties of these configurations. The configurations considered include the real Pappus configuration, a real configuration of twelve points and sixteen lines bearing the symbol $12_4, 16_3$ and another comprising thirty-six points and eighty-four lines bearing the symbol $36_7, 84_3$. (Received February 15, 1935.)

152. Professor E. T. Bell: *A detail in Kronecker's program.*

A misunderstanding regarding the freedom from circularity of Kronecker's method for eliminating all but the positive integers from analysis has recently been repeated. It is shown that the misunderstanding is removable, so far as negative numbers are concerned, by the correct form of the division transformation, not explicitly given by Kronecker, which is due to Wedderburn. Similar remarks apply to algebraic numbers. The paper will appear in the Journal of Philosophy of Science. (Received March 7, 1935.)

153. Mr. Edwin W. Paxson: *An abstract triplet metric space.* Preliminary report.

The usual concepts and postulates for vector spaces are taken with the replacement, however, of the set for the norm by a set of postulates for a function of three places. A trivial example is the space of ordered number-pairs in the plane, with the area of triangles as the function above, $N(x, y, z)$. With

suitable definitions for N, the spaces of real continuous functions of real variables, and vector spaces, are other such systems. A triplet of elements is considered as the space unit of structure, and sequences of these are studied, as are functions on the triplet space to that space. (Received March 7, 1935.)

154. Miss Dorothy Manning: *On simple transitive permutation groups with transitive Abelian subgroups of the same degree.*

It has been shown by I. Schur that a simply transitive permutation group of degree n (n not a prime) in which there is a circular permutation of degree n is imprimitive and compound. It can also be proved that if a simply transitive permutation group G of degree p^{a+b} (p a prime) has a transitive Abelian subgroup of degree p^{a+b} and type (a, b), G is imprimitive and compound unless $a=b$. There exists a simply transitive primitive group of degree p^{2a} with a transitive Abelian subgroup of order p^{2a} and type (a, a) provided $p^{2a}>4$. (Received March 9, 1935.)

155. Professor H. F. Blichfeldt: *On geometry of numbers.*

The author has found that a slight modification of Winkowski's fundamental geometrical method gives surprising accuracy in certain applications. Special attention is given to the simultaneous approximation to two irrational numbers in rational fractions with common denominator, new lower limits being obtained. (Received March 9, 1935.)

156. Professor J. V. Uspensky: *A simple proof of Tschebysheff's inequalities.*

A method used in proving a certain fundamental lemma in the author's article, *On an expansion of the remainder in the Gaussian quadrature formula* (this Bulletin, vol. 40 (1934), p. 871) gives also a natural and simple proof of some important inequalities due to Tschebysheff. (Received March 9, 1935.)

157. Professor C. H. Forsyth: *A fallacy of the so-called osculatory interpolation formulas.*

Professor Forsyth shows that Newton's formula, which is usually the formula modified or used as a base to obtain the osculatory interpolation formula, is osculatory itself. (Received March 9, 1935.)

158. Mr. I. E. Highberg, Professor A. D. Michal, and Mr. A. E. Taylor: *The geometry of abstract euclidean spaces.*

An abstract euclidean space E (see A. D. Michal, abstract 40-11-383) is a normed linear space in which there is defined a Hermitian inner product (x, y), which is a continuous function on E^2 to C. This inner product is used to build up the *geometry* of E, (angle, distance, length of arc, etc.). The general theory of motions and rotations in E is considered. A rotation is shown to be additive, homogeneous of degree one for real multipliers, and, in case E is a complete space, continuous. The theory of Lebesgue integrals is discussed, together with its bearing on rectifiability of curves in E. For a Hilbert space or for an n-dimensional euclidean space, a necessary and sufficient condition that a

curve be rectifiable is that its defining function be of bounded variation. (Received March 9, 1935.)

159. Mr. I. E. Highberg and Mr. A. E. Taylor: *Sets of independent postulates for linear spaces and vector spaces.* Preliminary report.

We have obtained a set of eleven postulates, defining a linear vector space closed under multiplication of reals. Equality is introduced on a postulational basis, and the reflexivity, symmetry, and transitivity properties follow as theorems. It is known that nine of these postulates are independent. Other systems of postulates are considered, dealing with linear spaces (non-metric) and with vector spaces closed under multiplication by complex numbers. In, these systems we have not stated the postulates for the multiplier domain (the system of real or complex numbers). In a later paper we intend to consider systems in which the multiplier domain is an abstract field, in which case the independence of all the postulates will be considered. (Received March 9, 1935.)

160. Mr. A. E. Taylor: *A set of independent postulates for a metric space.*

A metric space is herein defined by a set of six independent postulates. The equivalence relation in the space is a postulated relation, characterized by two of the six postulates. We are then able to show that it has the usual properties of equality. (Received March 9, 1935.)

161. Mr. Victor Elconin: *A theorem relating the differentials of Fréchet and Gateaux.*

Let $f(x)$ be in a Banach space for all x in some neighborhood of an element a in a Banach space E. It is well known that the Gateaux differential of $f(a)$ exists equal to the Fréchet differential of $f(a)$ if the latter exists. The theorem of this note states the converse of this proposition under the hypotheses that (i) some neighborhood of a is compact, and (ii) the Gateaux differential of $f(x)$ exists, continuous in x, linear in the increment of x, for all x in some neighborhood of a. Known results for n-space, Hilbert space, and certain function-spaces are shown to be instances of the theorem. (Received March 9, 1935.)

162. Dr. R. D. James and Mr. H. S. Zuckerman: *New results for the number $g(n)$ in Waring's problem.* Preliminary report.

The number $G(n)$ in Waring's problem is defined as the least value of s such that all *sufficiently large* integers are sums of s integral nth powers $\geqq 0$. The number $g(n)$ is defined as the least value of s such that *all* positive integers are sums of s integral nth powers $\geqq 0$. In the Comptes Rendus, Academie des Sciences, URSS, May, 1934, Vinogradow states that he has proved that $G(n) < 3k^2 + 4n$, k an integer, $k = n \log 8n + \theta$, $0 < \theta < 1$, and gives an outline of his proof. In this paper we supply a complete proof. At the same time we determine the value of C such that all integers $\geqq C$ can be expressed as a sum of $G(n)$ integral nth powers $\geqq 0$. As an illustration we consider 15th powers and

are able to prove that $g(15)=33203$. This is the first exact value of $g(n)$ to be found since it was shown that $g(2)=4$ and $g(3)=9$. We hope to be able to prove a general result for $g(n)$. (Received March 9, 1935.)

163. Professor B. A. Bernstein: *Postulates for Boolean algebra involving the operation of complete disjunction.*

The operation of complete disjunction 0, given by $a0b=ab'+a'b$, has important properties. The author gives a set of ten postulates for Boolean algebra in terms of 0, $\times$, $'$, and obtains from these postulates some of the properties of 0. The postulates, like Sheffer's postulates in terms of "rejection," contain no postulated special elements, and no existence proposition other than one stating that the number of elements is at least two. A slight change in the postulates transforms them into a set of postulates for the logic of propositions. (Received March 9, 1935.)

164. Professor A. R. Williams: *Correspondences connected with a pencil of n-ics.*

When $n(n+3)/2-2$ base points of a pencil of n-ics are fixed in general position, and another describes a curve of order m, the locus of the remaining $(n-1)(n-2)/2$ is a curve of order $m(n^2-1)$ which has an mn-fold point at each of the base points. If $nq-(q-1)(q-2)/2$ of the fixed points are on a curve of order q, the number of variable points is $(n-q)(n+q-3)/2$, and their locus is a curve of order $m(n^2-q^2-1)$ which has an $m(n-q)$-fold point at each of the fixed points on the q-ic. Other features of the correspondence are discussed. (Received March 11, 1935.)

165. Professors H. E. Bray and G. C. Evans: *A simplification in the theory of weak convergence.*

By means of the correspondence between additive functions of point sets and additive functions of boundaries, the fundamental theorem on the weak convergence of a subsequence of a sequence of additive functions of point sets, uniformly bounded in total variation, may be reduced to the Cantor diagonal process for functions of finite plurisegments. (Received March 11, 1935.)

166. Professor Tibor Radó: *The isoperimetric inequality on the unit sphere.*

Let C be a simple closed curve of length L, located on the unit sphere, and let A denote the area of any one of the two regions bounded by C on the unit sphere. Then $(2\pi-A)^2-L^2\geqq 4\pi^2$, and the sign of equality holds if and only if C is a circle. F. Bernstein (Mathematische Annalen, vol. 60 (1905), pp. 117–136) proved this theorem under the restriction that the curve C can be covered by a hemisphere. The purpose of the present paper is to establish the theorem in complete generality. The generalization depends upon the following lemma: if the simple closed curve C, located on the unit sphere, has a length less than 2π, then C can be covered by an open hemisphere. (Received February 16, 1935.)

167. Professor I. S. and Dr. E. S. Sokolnikoff: *A Dirichlet problem for an ellipsoid lying between two infinite planes.*

A solution of the problem of determining the electrical field of a sphere lying between two infinite conducting planes was published by C. M. Rigby in Proceedings of the London Mathematical Society, (2), vol. 33 (1932). The present paper treats the more general problem in which the sphere is replaced by an ellipsoid of revolution. It contains Rigby's solution as a special case, and also enables one to calculate the field due to a charged circular disk or a thin rod placed between two parallel infinite conducting planes. The calculation of the harmonic function is reduced to the solution of an infinite system of linear algebraic equations, which are treated by a method of successive approximations. (Received March 1, 1935.)

168. Professor W. E. Roth: *On the characteristic values of the matrix $f(A, B)$.*

Let $A=A_1+A_2+\cdots+A_r$ be a given matrix of order n, where $|A_i-\lambda I| =(a_i-\lambda)^{n_i}$, $(i=1, 2, \cdots, r)$, $a_i\neq a_j$, $i\neq j$; and let $B=(B_{ij})$, where B_{ij} are $n_i\times n_j$ matrices. In order that every polynomial $f(A, B)$ in A and B have all its characteristic values in the set $f(a_i, b_j)$, where b_j are the characteristic values of B, then necessarily $|B|=\Pi_{i=1}^{r}|B_{ii}|$ and the characteristic value of B associated with a_i are those and only those of B_{ii}. If A is given in the Jordan canonical form, the matrix B is then displayed so that the polynomials $f(A, B)$ have the property above. The procedure of this paper is independent of that folllowed by Bruton. (Thesis, 1932, University of Wisconsin. Abstract, this Bulletin, No. 38-9-196.) (Received February 13, 1935.)

169. Dr. Ralph Hull (National Research Fellow): *Maximal orders in rational cyclic division algebras of odd prime degree.*

Let A be a cyclic division algebra of odd prime degree n over the field R of all rational numbers. Then A has a cyclic generation $A=(\sigma, Z, S)$, where Z is a cyclic field of degree n over R with a prime conductor $p\equiv 1 \pmod n$, discriminant p^{n-1}, and $\sigma=q_1\cdots q_s$ $(s\geqq 2)$, where the q are distinct primes, each an n-ic non-residue modulo p, but such that σ is an n-ic residue modulo p. (See abstract 40-3-128, March, 1934.) We have $A=Z+uZ+\cdots+u^{n-1}Z$, where $zu=uz^{S}$ for every z in Z and $u^n=\sigma$. There exists an integral quantity α of Z such that $N_{ZR}(\alpha)=p^{n-1}$. It is proved that there are exactly n distinct maximal orders (integral sets) in A which contain all the integers of Z and the quantity u. These correspond to the n solutions λ of $\lambda^n\equiv\sigma \pmod p$ and for a fixed λ the corresponding maximal order has a basis y^iz_k $(i=0, \cdots, n-1; j=1, \cdots, n)$, where the z_k form a normal basis of the integers of Z and $y=(\lambda-u)\alpha/p$. These results are similar to those obtained by Albert (this Bulletin, vol. 40 (1934), pp. 164–176) for rational generalized quaternion algebras, but the methods used are more general, involving Hasse's theory of the invariants of cyclic algebra (Transactions of this Society, vol. 34 (1932), pp. 171–214), the well known representation of A as an algebra or matrices with elements in Z, and a study of the π-components of certain orders in A for all rational primes π. (Received March 7, 1935.)

170. Dr. Solomon Kullback: *On the distribution problem of statistics.*

We consider the problem of finding the distribution function (or probability law) of the functions $\mu_r(x_1, x_2, \cdots, x_n)$ $(r=1, 2, \cdots, p)$ of the variables $x_1, x_2, \cdots, x_n$ whose distribution function is given by the monotone absolutely additive set function $\phi(E)$ (where $F(x_1, x_2, \cdots, x_n)$ is the corresponding point function) such that $0 \leqq \phi(E) \leqq 1$ and $\phi(S)=1$ where S is the entire $(x_1x_2 \cdots x_n)$-space. The functions μ_r are real, single-valued, and the set of values of μ_r may be finite, a denumerable infinity, or form a continuum. By means of E. K. Haviland's extension to more than one dimension of P. Lévy's inversion formula for Fourier-Stieltjes transforms we are enabled to express the characteristic function of the distribution function of $\mu_1, \mu_2, \cdots, \mu_p$ as an integral of known functions over the entire $(x_1x_2 \cdots x_n)$-space. The theorems of this paper are generalizations of results previously obtained by C. V. L. Charlier, T. Kameda, and the author. (Received February 15, 1935.)

171. Dr. R. E. Basye: *Concerning an intrinsic property of plane continua.*

Let H and K be two mutually exclusive closed subsets of a compact plane continuum M. If for each point A of H and each point B of K there exists a finite number of subcontinua of M whose sum separates A from B in M, then there exists a finite number of subcontinua of M whose sum separates H from K in M. An example of a continuum showing that this result does not hold in three dimensions was obtained some years ago by W. T. Reid and presented by him at the meeting of the Society on February 23, 1935. (Received February 16, 1935.)

172. Dr. W. R. Thompson: *On criteria for rejection of observations and the distribution of the ratio of deviation to sample standard deviation.*

If one draws from a normal population a random sample of N observations $\{x_i\}$ whose mean is $\bar{x}$ and $N \cdot s^2 = \sum (x_i - x)^2$, then it is proved in this paper that the distribution of $\tau \equiv (x-\bar{x})/s$ in repeated samples of the same size is given by substitution of $(n)^{1/2} \cdot z = t = (n)^{1/2} \cdot \tau / (n+1-\tau^2)^{1/2}$ in the z or t distribution of "Student" and R. A. Fisher, where $n=N-2$, i.e., $\tau = (n+1)^{1/2} \cdot \sin \alpha$, for $(n)^{1/2} \cdot \tan \alpha = t$. The probability, P, that $|\tau| > \tau_0$ is the same as that $|t| > (n)^{1/2} \cdot \tau_0 / (n+1-\tau_0^2)^{1/2}$. Rejection criteria are readily established on a variety of bases, e.g., $P=\phi/N$, where ϕ is constant, and convenient tables are formed readily from those of "Student." Accordingly, all observations deviating by more than $s \cdot \tau_0$ from the sample mean are rejected. If one generalizes $\tau \equiv (\bar{x}_k - \bar{x})/s$, where $\bar{x}_k$ is the mean of k arbitrary elements of the sample of N, then it is proved that the distribution of τ is given by that of t by substitution of $t = \tau \left[(n \cdot k)/(n+2-k-k \cdot \tau^2) \right]^{1/2}$; i.e., $(k)^{1/2} \cdot \tau = (n+2-k)^{1/2} \cdot \sin \alpha$. (Received March 4, 1935.)

173. Professor G. D. Birkhoff and Dr. M. R. Hestenes:

Natural isoperimetric conditions for variable end point problems in the calculus of variations.

Consider the accessory minimum problem associated with the second variation $J_2(\eta)$ in the variable end point case. The minimum number m of natural isoperimetric conditions needed to reduce the arc $(\eta) \equiv (0)$ to a minimizing arc will be called the type number m of J_2. Let h be the maximum number of linearly independent accessory extremals satisfying the end conditions and having $J_2 < 0$ for every proper linear combination of these extremals. Let k be the maximum number of linearly independent accessory extremals having $\eta_i(x_1) = \eta_i(x_2) = 0$ and which are such that no proper linear combination of these extremals satisfies the accessory transversality conditions. The number $r = h + k$ will be called the order of concavity of J_2. Let s denote the sum of the orders of the conjugate points of x_1 between x_1 and x_2. If the strengthened condition of Legendre holds, then $m = r + s$, even in the degenerate case. In the case of one variable end point, the type number m of J_2 is equal to the sum of the orders of the focal points between x_1 and x_2. In the general variable end point cases consider an extremal E satisfying the end conditions and the transversality conditions. By the type number m' of E will be meant the minimum number of isoperimetric conditions needed to reduce E to a weak minimizing arc. If the strengthened condition of Legendre holds and E is not degenerate, then $m' = m$, where m is the type number of J_2 on E. (Received February 18, 1935.)

174. Dr. Borge Jessen and Dr. Aurel Wintner: *Distribution functions and the Riemann zeta function.*

The paper starts with a systematic study of distribution functions in k-dimensional space and in particular of their infinite convolutions representing, in the language of the calculus of probability, the distributions arising by addition of an infinite number of independent random variables. The results are applied to almost periodic funtions and in particular to the Riemann zeta function. (Received February 26, 1935.)

175. Dr. L. M. Blumenthal (National Research Fellow): *Concerning metric transforms of metric spaces.*

Let $\phi(x)$ be a real function defined for every value $x = pq$, where pq is the distance of two points p, q, of a metric space M. We say that $\phi(x)$ is *metrically defined* over the space M. A space M' is called the *metric transform* of M by $\phi(x)$ provided (1) $\phi(x)$ is metrically defined over M, (2) the points of M and M' are in one-to-one, reciprocal, correspondence, and (3) if p', q' are points of M' corresponding to p, q of M, respectively, then $p'q' = \phi(pq)$. We write $M' = \phi(M)$. Consider the class Σ of monotonic increasing *concave* functions $\sigma(x)$ (i.e., for $x_1 < x < x_2$, the three-rowed determinant in which the ith row is $\sigma(\xi_i)$, ξ_i, 1 and ξ_1, ξ_2, ξ_3 are respectively x_1, x, and x_2, is greater than zero) such that $\sigma(0) = 0$. The following theorems are proved: Theorem 1. *If M is a metric space and $\sigma(x)$ is metrically defined over M, then $M' = \sigma(M)$ is a metric space.* Theorem 2. *If M is a metric space and $\sigma(x) = x^k$, $0 \leqq k \leqq 1/2$, then $M' = \sigma(M)$ has each quad-*

ruple of its points congruent with four points of the three-dimensional euclidean space R_3. It is shown that $k=1/2$ is the greatest value of k for which Theorem 2 is valid. It is observed that the transform M' of a metric space M by a function $\sigma(x)$ of the class Σ is never *convex*. (Received February 21, 1935.)

176. Dr. J. F. Randolph: *Some density properties of point sets.*

The study of density functions began soon after the initial work on measure for point sets. Lebesgue himself showed that at almost all points of a Lebesgue measurable set the density of the set exists and is unity. The situation for the density functions of plane point sets in relation to Carathéodory linear measure is much more complicated, and considerable work has been done along this line. Much of this work has been concerned with the behavior of the density functions of a set at points of the set itself. R. L. Jeffery (*Sets of k-extent in n-dimensional space*, Transactions of this Society, vol. 35 (1933), pp. 629–647) considered the behavior of the density functions at points of the complement of the set as well, and obtained some enlightening results. In this paper, by entirely different methods, these results of Jeffery's are obtained and extended. Special cases of these results reveal new properties of Lebesgue measure for sets on a straight line. (Received March 5, 1935.)

177. Dr. Rufus Oldenburger: *Characteristic numbers of trilinear forms.*

Characteristic numbers of a trilinear form are defined by means of three-way determinants, which can be expanded according to various signancies of the indices giving as many as $3n$ characteristic numbers for a trilinear form of order n. Relations between the ranks of a trilinear form and the existence of characteristic numbers as well as their multiplicity are obtained. It is further shown that characteristic numbers are invariant under the class of non-singular linear transformations which leave unchanged a form of the type $x_iy_iz_i$, $i=1, 2, \cdots, n$. (Received March 9, 1935.)

178. Professor R. L. Wilder: *On locally connected spaces.*

Let S be a locally compact metric space that is locally i-connected, for $0 \leqq i \leqq k$ (where k is a non-negative integer), in terms of Vietoris cycles and chains mod $m \geqq 2$. If F is a self-compact subset of S, and U an open subset of S containing F, then at most a finite number of i-cycles of F are independent with respect to homologies in U. As a corollary of this result, if S is compact, the Betti numbers $p^i(S, m)$ are all finite; and if in addition "small" $(k+1)$-cycles bound in S, then $p^{k+1}(S, m)$ is finite. Consequently if M is a g. c. n-m. (Annals of Mathematics, vol. 35, p. 878), all its Betti numbers are finite. Supplementary results include (1) a lemma characterizing absence of local i-connectedness (for $i=0$ proved by Moore; see this Bulletin, vol. 29, p. 296), and (2) the following theorem: In $E_n (n \geqq 2)$, let M be a compact continuum which is locally i-connected for $0 \leqq i \leqq n-2$. Then if D is a domain complementary to M, the boundary of D is a Jordan continuum (for $n=2$, proved by M. Torhorst, Mathematische Zeitschrift, vol. 9, pp. 44–65). (Received March 9, 1935.)

179. Mr. H. E. Vaughan, Jr.: *Local Betti numbers.*

To each point a of a compact metric space M at which $\dim_a M = n$ a set of integers $\beta^i(a, M)$, $0 \leqq i \leqq n-1$, is assigned as follows: $\beta^i(a, M)$ is the least integer m, if it exists, such that for every $\epsilon > 0$ there exists a neighborhood of a of diameter less than ϵ whose boundary is $(n-1)$-dimensional and has m as its ith Betti number. If no such integer exists we define $\beta^i(a, M) = \omega$ or κ_0 in an obvious manner. Using these numbers we define *edge* and *kernel* points of M. The following characterizations are obtained: Let M be a 2-dimensional, locally 1-connected, closed cantorian manifold such that, if a is any point of M, $\beta^1(a, M) \leqq 1$. Then M is a 2-manifold. Let K be a 2-dimensional locally 1-connected irreducible membrane with respect to a 1-cycle carried by a simple closed curve J such that $p^1(K) = p^2(K) = 0$ and, if a is any point of K, $\beta^1(a, K) \leqq 1$ while if a is, in particular, a point of J, $\beta^1(a, K) = 0$. Then K is a closed 2-cell. By replacing the condition $p^1(K) = 0$ in the preceding theorem by $p^1(K) = 1$, a characterization of the Möbius strip is obtained. Similar changes give characterizations of other simple surfaces. (Received March 9, 1935.)

180. Professor G. D. Birkhoff and Dr. M. R. Hestenes: *Boundary value problems and the calculus of variations.*

Let $J(\eta, \sigma)$ be a quadratic functional in η_i, η_i' continuous in σ and subject to end conditions $\Psi_\mu(\eta, \sigma) = 0$ linear in the end values η_{i1}, η_{i2} and continuous in σ. This functional determines a boundary value problem. Suppose the strengthened condition of Legendre holds for all σ. If σ is a characteristic root let $d(\sigma)$ be its order. Otherwise set $d(\sigma) = 0$. Let $m(\sigma)$ be the type number of $J(\eta, \sigma)$ (see abstract 41-3-173). For σ near σ_0 we have $m(\sigma) \geqq m(\sigma_0)$, and if $d(\sigma) = 0$, $(\sigma \neq \sigma_0)$, then $m(\sigma_0) \leqq m(\sigma) \leqq m(\sigma_0) + d(\sigma_0)$. There are at least $|m(\sigma_2) - m(\sigma_1)|$ characteristic roots on the closed interval $\sigma_1\sigma_2$. If for every integer k there exists a constant σ_k and k arcs η_{ia} satisfying $\Psi_\mu(\eta, \sigma_k) = 0$ and having $J(\eta_a b_a, \sigma_k) < 0$ for every set $(b) \neq (0)$, then there exist infinitely many characteristic roots. This criterion is necessary in case $m(\sigma)$ is such that $m(\sigma) + d(\sigma) \leqq m(\sigma')$, $(\sigma < \sigma')$. In this case there are exactly $m(\sigma_2) - m(\sigma_1)$ characteristic roots on $\sigma_1 \leqq \sigma < \sigma_2$. If the end conditions are independent of σ and $J(\eta, \sigma) > J(\eta, \sigma')$ for $\sigma < \sigma'$ and $(\eta) \not\equiv (0)$, then $m(\sigma) + d(\sigma) \leqq m(\sigma')$, $(\sigma < \sigma')$; this case is the one treated by Morse in his Colloquium Lectures, Chapter IV. These and further results are immediate consequences of the theory of natural isoperimetric conditions. (Received March 6, 1935.)

181. Professor R. L. Wilder: *On free subsets of E_n.*

Earlier results for the plane and ordinary 3-space by the same author (Fundamenta Mathematicae, vol. 21, pp. 156–167), relating to a question raised by Borsuk, are extended to higher dimensions. In $E_n (n > 2)$, let C be a free, compact continuum which cuts E_n, is locally i-connected for $0 \leqq i \leqq n-3$, and for which $p^1(C)$ (or $p^{m-2}(C)$) is finite. Then C is a generalized closed $(n-1)$-manifold (Annals of Mathematics, vol. 35, p. 878). As a locally contractible continuum is locally i-connected, an obvious corollary to the foregoing is obtained. When chains mod $m \geqq 2$ are employed, the conclusion of the above theorem follows if C is only assumed locally i-connected for $0 \leqq i \leqq (n-1)/2$, with

no restriction on the Betti numbers. If a free, compact continuum which cuts E_n is deformable without meeting itself, then without further assumptions it is a g. c. $(n-1)$-m. Other theorems regarding particular types of transformations are also obtained. (Received March 9, 1935.)

182. Professor Walther Mayer: *The differential geometry of the submanifolds of the R_n of constant curvature.*

The present work is a new representation of the theory of curvature of submanifolds of spaces of constant curvature developed by C. Burstin and the author. (Received March 9, 1935.)

183. Dr. E. F. Beckenbach: *On subharmonic functions.*

It is shown that an equivalent form of a theorem of Montel and Radó is: if $p(u, v)$ is a positive continuous function, then a necessary and sufficient condition that $\log p(u, v)$ be subharmonic is that for every analytic function $f(u+iv)$ the function $p(u, v)\,|f(u+iv)|$ be subharmonic. Other equivalent forms are obtained. Applications are made to surfaces of negative curvature and to minimal surfaces. (Received March 7, 1935.)

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OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

STANFORD UNIVERSITY, April 6, 1935.

Abstracts must be in the hands of Associate Secretary T. M. Putnam, University of California, Berkeley, California, not later than March 9. All these will appear in the March issue of this Bulletin.

NEW YORK CITY, April 19-20, 1935.

Abstracts must be in the hands of Associate Secretary J. R. Kline, Low Memorial Library, Columbia University, New York City, not later than March 23. Abstracts received by the Secretaries on or before March 9 will appear in the March issue of this Bulletin. There will be a symposium on the topic *The problem of three bodies, and allied problems*, at which the speakers will be Professors G. D. Birkhoff, E. W. Brown, and Aurel Wintner.

CHICAGO, ILLINOIS, April 19-20, 1935.

Abstracts must be in the hands of Associate secretary M. H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than March 23. Abstracts received by the Secretaries on or before March 9 will appear in the March issue of this Bulletin. By invitation of the Program Committee, Professor L. M. Graves will deliver an address on *Topics in functional calculus*.

R. G. D. RICHARDSON, *Secretary of the Society.*

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

Subscriptions to the BULLETIN, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to the American Mathematical Society, 450-459 Ahnaip St., Menasha, Wis., or Low Memorial Library, Columbia University, New York.

The initiation fees and the annual dues of members of the Society (see this BULLETIN, p. 322, May, 1930; and the List of Officers and Members, September, 1934, p. 66), are payable to the Treasurer of the Society, Professor G. W. Mullins, Low Memorial Library, Columbia University, New York City.

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